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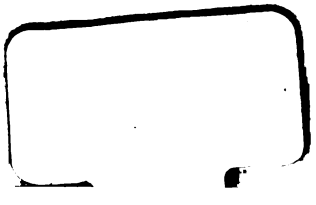

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Solved Questions.

2357. (T. Cotterill, M.A.) — (1) If the determinant $\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}$ vanish, the equations

$$a(by^2+cy^2-2dyz) = \beta(cx^2+az^2-2ezx) = \gamma(ay^2+bx^2-2fxy)$$

are satisfied by two conjugate pairs of values of the variables, one pair being independent of the constants α, β, γ. Find the equation to the quadric satisfying the values $y = z = 0$, $z = x = 0$, $x = y = 0$, and the last pair of values, and the linear equations satisfying each conjugate pair. (2) Show that such a system of equations is the analytical representation of the projection of the three circles described on the lines connecting the opposite points of intersection of four lines in a plane as diameters, the circle circumscribing their diagonal triangle, and the line at infinity 54

2439. (T. Cotterill, M.A.)—If a, b, c, d are collinear points as well as x, y, z, t, and in the same plane, then, of the sixteen intersections of ax, ay, bx, by with cz, ct, dz, dt, eight lie on one conic and eight on another. The four points of intersection of these conics lie on a third conic through a, b, x, y and a fourth through c, d, z, t, and these conics are respectively harmonics to the lengths (ab, xy) and (cd, zt). Also the tangents to the four conics at any point of intersection are harmonic 59

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2664. (T. Cotterill, M.A.) — (1) Five points (no three in the same line, and no four in the same plane) determine, by the lines and planes through them on a plane, a system of ten points and ten lines, the points lying in threes on the lines and the lines passing in threes through the points (Cayley). Show that the figure is its own polar reciprocal to a conic; and that, if a conic and triangle in its plane are given, the rest of the figure can be constructed. (2) A quadric through the five points cuts the plane in a conic containing triangles conjugate to the fixed conic. There is a quadric passing through the fixed conic, to which one of the five points and the plane are pole and polar, the remaining four points forming a self-conjugate tetrahedron 93

9019. (Asparagus.)—In a given ellipse is inscribed a triangle ABC of maximum perimeter ($2s$): prove that, if ρ_1, ρ_2, ρ_3 be the radii of curvature of the ellipse at A, B, C,

$$\rho_1 \cos^3 \tfrac{1}{2}A = \rho_2 \cos^3 \tfrac{1}{2}B = \rho_3 \cos^3 \tfrac{1}{2}C = s, \quad \rho_1 \rho_2 \rho_3 = 8r^3,$$

r being the radius of the circle inscribed in the triangle ABC 65

9807. (Professor G. B. M. Zerr.) — The perpendiculars from the vertices of a triangle upon the central axis (the line which passes through the circum-centre, the orthocentre, the nine-point centre, and the centroid) are proportional to

$$\cos A \sin (B - C), \quad \cos B \sin (C - A), \quad \cos C \sin (A - B),$$

those on one side of the line being reckoned positive, and those on the other negative 31

10596. (Professor Déprez.) — Soient a, b, c les côtés d'un triangle sphérique ABC; r le rayon du cercle inscrit; x, y, z les distances d'un point de ce cercle aux côtés de ABC. Démontrer la formule

$$\cos \tfrac{1}{2}A (\sin y \sin z)^{\frac{1}{2}} + \ldots = 2 \tan r \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \ldots\ldots 63$$

10693. (Professor Lampe, LL.D.) — Find all the values of x that satisfy the equation $64 \sin^2 x \sin^2 2x \sin^2 3x = 7$ 91, 102

10730. (Professor Déprez.) — Trouver l'enveloppe des axes des coniques touchant les côtés AB, AC d'un triangle, et ayant pour asymptote le troisième côté BC 73

10790. (H. W. Segar, B.A.—*Corrected.*)—Show that

$$\tfrac{1}{4}\pi \int_0^{\frac{1}{2}\pi} \log (1 + \cos\theta \cos x) \frac{dx}{\cos x} = \frac{\cos\theta}{1^2} - \frac{\cos 3\theta}{3^2} + \frac{\cos 5\theta}{5^2} - \ldots \ldots 26$$

10825. (W. J. C. Sharp, M.A.)—Prove (1), (2), and therefrom deduce (3), (4), where

$$(-1)^m (2 \sin A)^m \sin (mA + B + \tfrac{1}{2}m\pi)$$
$$= \sin B - \frac{m}{1} \sin (2A + B) + \frac{m(m-1)}{1 \cdot 2} \sin (4A + B) - \ldots \ldots (1)$$

$$-1)^m (2 \sin A)^m \cos (mA + B + \tfrac{1}{2}m\pi)$$
$$= \cos B - \frac{m}{1} \cos (2A + B) + \frac{m(m-1)}{1 \cdot 2} \cos (6A + B) - \ldots \ldots (2);$$

$$\int \sin^m x \sin (m+n)x \, . \, dx, \quad \int \sin^m x \cos (m+n)x \, . \, dx \ldots\ldots (3, 4)$$
..... 109

10860. (H. J. Woodall.)—Trace the curve

$$(a - x^2)(x^2 + y^2)^3 = 4x^2 [(a - x)^2 + y^2]^2 \ldots\ldots 106$$

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11039. (Professor Sylvester.) — Let $x_1, x_2, \ldots, x_n$ represent $x+a_1$, $x+a_2, \ldots, x+a_n$, where $a_1, a_2, \ldots, a_n$ are any *real* quantities. Also let $b_1, b_2, \ldots, b_n$ be any positive quantities. Prove that the continued fraction

$$x_1 - \frac{b_1}{x_2 -} \frac{b_2}{x_3 -} \cdots \frac{b_{n-1}}{-x_n}$$

changes sign exactly $2n-1$ times as x passes from one end of infinity to the other 41

13986. (Professor Nanson.)—If $t_1, t_2, \ldots, t_m$ are the m tangents and $n_1, n_2, \ldots, n_{m+1}$ are the $m+1$ normals from any point to the curve $y^m = ax^{m-1}$, prove that $m^m n_1 n_2 \ldots n_{m+1} = a t_1 t_2 \ldots t_m$ 70

15079. (Rev. T. Roach, M.A.)—Prove geometrically

$$\sin^{-1}\tfrac{4}{5} + \sin^{-1}\tfrac{5}{13} + \sin^{-1}\tfrac{16}{65} = \tfrac{1}{2}\pi \quad \ldots\ldots 105$$

15274. (A. J. Murphy.) — I, I_a, I_b, I_c are the centres of the in- and ex-circles of a triangle ABC, and these circles touch the side BC in X, X_a, X_b, X_c respectively. The internal and external bisectors of the angle A meet the circum-circle in T and S respectively, and SI, SI_a, TI_b, TI_c meet the circum-circle in L, L_a, L_b, L_c respectively. Show that LX, L_aX_a, L_bX_b, and L_cX_c meet on the circum-circle and determine the position of the point of concurrence. Show also that L, L_a, L_b, L_c are the points of contact of the circum-circle with the four circles that can be described to touch the circum-circle and the sides AB, AC 84

15478. (Professor Nanson.)—If r, θ are the focal polar co-ordinates of the foot of a normal from r', θ' to a conic, prove that

$$r_1 r_2 r_3 r_4 = b^2 e^{-2} r'^2, \quad \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\theta' + (2n+1)\pi \ldots\ldots 28$$

15495. (The late Rev. Prebendary Whitworth, M.A.) — Show geometrically that, if ABCD be a quadrilateral inscribed in a circle, and if AP, AQ, AR be perpendiculars from A on BC, CD, BD respectively, then

$$BC/AP + CD/AQ = BD/AR \ldots\ldots 22$$

15523. (A. M. Nesbitt, M.A.)—A man, who has m hats of his own in his hall, is visited by n friends, each wearing a hat. They leave their hats with those of their host. When they are going away they are unfortunately not in a condition to distinguish between one hat and another. Find the chance that no guest takes away his own hat 49

15599. (A. M. Nesbitt, M.A.)—Two tangents drawn from a point T to a parabola meet any third tangent in F and F′. If the diameter through T meet the curve in P, prove that the tangent at P bisects FF′ 48

15621. (The late R. Tucker, M.A.)—ABC is a triangle. Ω (or Ω') is a B-point, and $\omega_1, \omega_2, \omega_3$ are the B-angles of the triangles ΩBC, ΩCA, ΩAB: prove that

$$\Sigma(\cot\omega) = \frac{1 + \cot A \cot B \cot C \cot\omega + \cot^2\omega + \cot^4\omega}{\cot\omega - \cot A \cot B \cot C} \ldots\ldots 17$$

15677. (Robt. W. D. Christie.—*Slightly modified.*) — Let p be any prime and the letters all *different* integers. Prove that

$$pqr = p^2 + p^2 s + lt + u = (LMN)^{\frac{1}{2}}, \ldots;$$

e.g., $103.18.35 = 25^2 + 25^2.53 + 25.635 + 265 = (630.3605.1854)^{\frac{1}{2}} \ldots$ 25

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15679. (Alexander Holm, M.A.) — In Diophantus v. 19 it is required to find three numbers such that, if each is subtracted from the cube of their sum, the remainder may be a cube.

Diophantus' method leads to the numbers

$$\frac{68574961}{549353259}, \quad \frac{134953109}{549353259}, \quad \frac{162707336}{549353259}.$$

In the *Ladies' Diary* for 1716–17 (Leybourn's *Math. Questions*, Vol. I., pp. 52–62) it is stated that the lowest numbers hitherto found to satisfy the problem are $\frac{13851}{85184}, \frac{18954}{85184}, \frac{19467}{85184}$; required to find still lower numbers, for example, $\frac{351}{3136}, \frac{832}{3136}, \frac{833}{3136}$ 98

15690. (The late R. Tucker, M.A.)—ABC is a triangle, K is the S. point, and ω_1, ω_2, ω_3 are the B. angles of the triangles BKC, CKA, AKB. Prove that $\Sigma \cot \omega_r = 6 \cot \omega - (\operatorname{cosec}^2 \omega + 2)/(2 \cot \omega)$ 29

15700. (Professor Nanson.)—Eliminate the variables from

$$L_1/M_1 = L_2/M_2 = \ldots = L_n/M_n$$

where L_p, M_p are linear homogeneous functions of $n-1$ variables ... 109

15713. (Professor Lauvernay.) — Dans un triangle rectangle ABC, α, β, γ sont les points de contact de cercles ex-inscrits respectivement avec les côtés BC, CA, AB. Démontrer que, si les droites $A\alpha$, $B\beta$, $C\gamma$ sont concourantes sur la circonférence inscrite à ce triangle, les côtés de ce triangle sont proportionnels aux nombres 3, 4, 5..... 21, 43

15721. (Professor M. W. Crofton, F.R.S.)—Two players A., B., whose chances of winning a game are p, q $(p+q=1)$, put down each a stake, and play on the condition that whoever wins two games running shall receive the whole. Show that in fairness their stakes ought to be in the proportion $p^2(1+q) : q^2(1+p)$ 24

15726. (Professor Sanjána, M.A.)—Show that the normals drawn at the extremities of any chord of a parabola and terminated by the axis have equal projections upon that chord; and that these projections are constant when the chord moves so that the algebraical difference of the ordinates of the two extremities has a constant projection on the chord. Prove also that this condition is satisfied by every focal chord..... 43

15729. (R. Tucker, M.A.)—The circle $dd'k$ touches AB, AC, and the arc BC of the circle ABC (internally); the circle $ee'k'$ touches BC, BA, and the arc CA; and the circle $ff'k''$ touches CA, CB, and the arc AB. Prove (i.) ed', fe', df' are parallel to AB, BC, CA respectively; (ii.) $Cd' . Ae' . Bf' = Af . Bd . Ce$; (iii.) ρ_1 (radius of circle $dd'k$) $= r \sec^2 \frac{1}{2}A$; (iv.) Ak, Bk', Ck'' intersect in a point 38

15733. (Professor M. W. Crofton, F.R.S.) — An urn contains two balls, A and B. The probability that A is black is p; that B is black, q. One ball is now drawn, and found to be black:—(1) What is the chance that this is A or B? (2) What is now the probability that A is black or B? (3) If the ball drawn is replaced in the urn, and a fresh drawing made, what is the chance it gives a black ball?

[The balls A, B may be supposed to have been taken from two bags containing black and white balls in given proportions] 62

15740. (S. C. Bagchi, B.A.)—Supposing that $\phi(\xi_1, \xi_2, \ldots, \xi_n)$ gives the most probable value of a magnitude which is observed to have values $\xi_1, \xi_2, \ldots, \xi_n$ for n different observations successively such that $\xi_2 - \xi_1 = \delta_2$,

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$\xi_2-\xi_1=\delta_2$, ..., $\xi_n-\xi_1=\delta_n$, show that a particular solution of the general differential equation for ϕ satisfies

$$n(\phi-\xi_1)=(n+1)\sum_{r=2}^{n}\delta_r \quad \ldots\ldots 92$$

15746. (The late R. Tucker, M.A.)—PQ is a chord of an ellipse, subtending a right angle at the vertex. Find the locus of the S.-point of the triangle PAQ 63

15748. (A. M. Nesbitt, M.A.)—A, B, K are three fixed points on a tangent to a conic, and the other tangents from A and B meet in C. Any straight line through K cuts the conic in P and Q, and AQ, BP meet in R. Prove that the locus of R is a conic inscribed in the triangle ABC 26

15749. (W. F. Beard, M.A. Suggested by Question 15599.)—With the figure of Question 15599, if the tangent at P meets TQ, TQ' at K, K' and FK', F'K meet at R, prove that TR is parallel to FF' 48

15751. (*Communicated by* J. A. H. Johnston, M.A.)—Two light rods OA, OB, of equal length l, are hinged at a fixed point O, and carry heavy rings of equal mass m at the ends A and B. A third light smooth rod runs through the two rings and is hinged at a point P, vertically above O, at a distance l' from it ($l'>l$). The whole is started from the configuration in which A and B are vertically above and below O respectively, by impressing an angular velocity ω upon the rod PAB. Show that when the angle AOB between the rods is 2θ the reaction at P is

$$[2ml(\omega^2-g/l')(l'^4-l^4)\cos\theta]/[l'^2-l^2\cos 2\theta]^2 \quad \ldots\ldots 17$$

15753. (Lt.-Col. Allan Cunningham, R.E.)—Solve in integers

$$\tfrac{1}{2}(q^2+1)=p^4 \quad \ldots\ldots 23$$

15755. (Professor Sanjána, M.A.)—Prove that the expansion of $\operatorname{sech}^{-1}x^{\frac{1}{2}}+\log x^{\frac{1}{2}}$ is

$$\log 2-\left(\frac{1}{2}\frac{x}{2}+\frac{1.3}{2.4}\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\frac{x^3}{6}+\ldots\right),$$

x lying between zero and unity 61

15758. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Show that the product of the two series

$$1^3+2^3+3^3+4^3+5^3+\ldots+n^3, \quad 1^5+2^5+3^5+4^5+5^5+\ldots+n^5$$

may be represented as a single series 58

15761. (Professor Genese.)—Prove that there always exists one point which has the same tetrahedral co-ordinates with respect to any two given tetrahedra 28

15763. (Professor Sanjána, M.A.)—If P, Q be any two points isogonally conjugate with regard to the triangle ABC, then

$$(AP.AQ)/(AB.AC)+(BP.BQ)/(BC.BA)+(CP.CQ)/(CA.CB)=1$$

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15764. (R. F. Davis, M.A.)—The tangent at A to the circum-circle of the triangle ABC meets the base BC produced in T. From T perpendiculars are drawn upon AB, AC, meeting the circum-diameter through A in P, Q respectively. Prove (geometrically) that P, Q are equidistant from the circum-centre 102

15766. (A. M. Nesbitt, M.A.)—The locus of the intersection of normals at the extremities of conjugate diameters of $x^2/a^2+y^2/b^2=1$ is the curve $2(b^2y^2+a^2x^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$ 28

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15768. (S. C. Bagchi, B.A.)—The rate of change of the Massieu-characteristic of a homogeneous mixture due to a small amount of one of its constituents being allowed to diffuse in it is invariantive with regard to the proportional increase in the molecular weight of each constituent. (Massieu-characteristic is called by Duhem the "thermodynamic potential at constant pressure") 77

15769. (R. Chartres.)—A random point being taken in each side of the triangle ABC, find the mean value of $p^{\lambda}q^{\mu}r^{\nu}$, x^n, and $\log x$. [Elementary proof required] 42

15770. (C. E. Youngman, M.A.)—A sum of money is made up of ordinary coins of the realm, each of as large value and as little frequency as may be consistent with the money being shared as follows by A., B., and C.:—first A. takes a penny and one-third of the rest, then B. a penny and one-third of the rest, then C. likewise, and the remainder they share equally. Then one of them discovers that his coins are worth a shilling each on the average. Find (without algebra) who this is..... 38

15771. (A. Holm, M.A.)—Find r rational numbers such that, if a given number is added to their sum or to the sum of any $r-1$ of them, the results may all be squares 30

15772. (R. W. D. Christie.)—Prove $A = B^2 + C^2$ (integers) when $A = \Sigma_0^n \{(m+n)!/[2n!(m-n)!]\}$ $(m \not> n)$.

E.g., $m = 5$, $n = 5, 4, 3, 2, 1, 0$; then

$$A = \Sigma_0^5 (1 + 9 + 28 + 35 + 15 + 1) = 8^2 + 5^2 \text{ ..... } 55$$

15773. (Rev. F. H. Jackson.)—Show that

$$\left(1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \ldots\right)\left(1 + \frac{x}{1^2} + \frac{x^2}{1^2.2^2.3^2} + \ldots\right)$$
$$= 1 - \frac{3!}{1^2.2^2.1^2}x^2 + \frac{6!}{1^2.2^2.3^2.4^2.1^2.2^2}x^4 - \ldots .$$

..... 32

15776. (Professor Hudson, M.A.)—Prove that the cusps of the evolute of the focus-roulette of a hyperbola on a straight line are at the same distance from that line 42

15777. (Professor Cochez.)—On donne $y^2 = x^3$. On demande le lieu des sommets des angles droits circonscrits à cet cubique 47

15779. (Professor Sanjána, M.A.)—Prove, by the elementary geometry of the conic sections, that the sum of the reciprocals of two perpendicular focal chords of a conic is of constant magnitude 58

15780. (D. Biddle.)—PQ is a chord of a given circle; LOM is an intercept of the ellipse which, having P, Q as its foci, passes through O, the centre of the circle, L, M being on the circumference. (1) Find the position of PQ when the equal triangles PQL, PQM are maxima; and their area, as such. (2) Consider similarly the cases in which the ellipse cuts the circumference in S, T as well, on the opposite side of P, Q, and in which (for the limited range) PQS, PQT exceed in area PQL, PQM, the corresponding triangles previously considered 72

15781. (Professor Neuberg.)—Soit A le sommet d'une parabole. En un point quelconque M de cette courbe on mène une corde MN perpendiculaire à MA. Trouver le lieu du milieu de cette corde 57

15782. (A. M. Nesbitt, M.A.)—The inscribed circle of a triangle ABC touches the sides AC, AB in E and F, and an escribed circle touches the same sides produced in E′ and F′. If the median AK cut EF in G and E′F′ in G′, prove that BGCG′ is a parallelogram 60

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15784. (R. F, Davis, M.A. Suggested by Question 15748.)—If ABCD be a quadrilateral whose opposite sides BA, CD meet in E, and any variable transversal EPQ meet AD, BC in P, Q respectively, then $AP.PD/EP^2 : BQ.QC/EQ^2$ is constant 39

15786. (I. Arnold.)—An equilateral triangle ABC is suspended at A and its side is kept vertical by a ring at B. It is required to find what the angular velocity of the triangle round AB must be in order that there may be no pressure at B 51

15787. (W. J. Dobbs, M.A.)—Show how to resolve a given force into three forces of given magnitudes acting respectively through three given points situated in one plane with the line of action of the given force 81

15791. (A. H. Bell.)—When is a triangular number a pentagonal number? Required, a general solution with examples 40

15792. (Professor H. Laughorne Orchard, M.A., B.Sc.)—Sum to n terms the series $1+24+162+640+1875+\ldots$ 67

15793. (Professor Nanson.)—Eliminate x, y, z, x', y', z' from

$$lx+my+nz=0, \quad lx'+my'+nz'=0;$$

$$(abcfgh)(xyz)^2=0, \quad (abcfgh)(x'y'z')^2=0, \quad (abcfgh)(xyz)(x'y'z')=0$$

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15794. (Robert W. D. Christie.)—Where p_n, q_n are any convergents of $p_n^2-2q_n^2=\pm 1$, prove

$$2\tan^{-1}\frac{q_n}{q_{n+1}} \pm \tan^{-1}\frac{1}{p_{2n+1}} = \tan^{-1}1 = \tfrac{1}{4}\pi \ldots\ldots (1),$$

$$2\tan^{-1}\frac{p_n}{p_{n+1}} \mp \tan^{-1}\frac{1}{p_{2n+1}} = \tan^{-1}1 = \tfrac{1}{4}\pi \ldots\ldots (2);$$

and show that Euler's and Machin's series are simple cases (signs alternate)..... 52

15795. (S. C. Bagchi, B.A.)—The picture of a system of latitude and longitude obtained by orthogonal projection on one of the polar-tangent planes is the same as that (on any plane) obtained by inversions from properly chosen centres 93

15797. (*Communicated by* M. V. A. Sastry, B.A.)—If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to a semi-cubical parabola $ay^2=x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$ 70

15799. (D. Biddle.)—From a point A on the circumference of a given circle a straight line of diameter's length is drawn in any direction in the same plane. (1) Find, on the circumference, another point P such that a straight line drawn from it through the centre shall meet the given line in X, making AX = AP. (2) Find the locus of X 79

15800. (R. F. Davis, M.A.)—Prove that the semi-axes of the ellipse which is inscribed in the triangle of reference and has for its centre the point whose trilinear co-ordinates are α, β, γ are determined by the equation $a\sqrt{(a^2-\rho^2)}+b\sqrt{(\beta^2-\rho^2)}+c\sqrt{(\gamma^2-\rho^2)}=0$ 44

15802. (*Communicated by* James Blaikie, M.A.)—Etant donnés deux points A et B, on prend les points I et I' qui partagent le segment AB dans un rapport donné α/β et l'on considère toutes les droites D telles que le rapport de leurs distances aux points A et B soit égal à α/β. Démontrer que les plans ID et I'D sont rectangulaires. En conclure:—(1) que les droites D qui passent par un point fixe S engendrent un cône

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dont les sections circulaires sont perpendiculaires aux droites SI et SI' ; (2) que les droites D qui sont situées dans un plan Π enveloppent une conique ayant pour foyers les projections des points I et I' sur le plan Π. Indiquer ce qui arrive lorsque le point S est situé sur la sphère qui a pour diamètre II' ou lorsque le plan Π est tangent à cette sphère. (See Guichard, *Compléments de Géométrie*) 85

15803. (Professor Neuberg.) — ABCD étant un rectangle donné, trouver sur le côté AD un point X tel que les volumes engendrés par le triangle ABX et le trapèze BCDX tournant autour de la droite BX aient un rapport donné k 72

15804. (Robert W. D. Christie.) — Let $p = m^4 - m^3 + m^2 - m + 1$. Prove $[(m^2+1)(m-1)]^{n+1} + m^{4n-1} = 0 \bmod p$ and generalize 60

15805. (A. H. Bell.)—Is 9462853 a prime number? 68, 86

15806. (Rev. F. H. Jackson.)—Given

$$P_n(x) = \frac{[2n]!}{(2)_n [n]! [n]!} \left\{ x^n - p^2 \frac{[n][n-1]}{[2][2n-1]} x^{n-2} + \dots \right\},$$

in which the p prefixes of the terms of the series form the sequence

$$(p^2, p^8, p^{15}, \dots, p^{r(r+2)}, \dots),$$

and $[n] = (p^n-1)/(p-1)$, $(2)_n = (1+p^n)(2)_{n-1}$, $[n]! = [n][n-1]!$, show that $\frac{[n]}{[2n-1]} \frac{P_n}{P_{n-1}}$ may be expressed as a continued fraction

$$\cfrac{[n]^2}{[2n+1][2n-1]} \frac{p^{n+2}}{x - \cfrac{[n+1]^2}{[2n+3][2n+1]} \frac{p^{n+3}}{x - \dots}}$$ 52

15811. (R. Chartres.) — From a variable point within a triangle perpendiculars are drawn to the sides. Find (1) the maximum value, (2) the mean value of the product of the perpendiculars, and (3) the mean value of the area of the triangle formed by joining the feet of the perpendiculars. [Elementary proof] 76

15818. (R. F. Davis, M.A.)—A triangle circumscribes a given circle (centre I, radius r) and has its orthocentre at a given point H. Prove that the angular points lie on a fixed conic whose focus is at H, and whose directrix KK' is found by producing IH to X, so that $IH.IX = 2r^2$, and drawing KXK' perpendicular to IHX 94

15819. (Professor Neuberg.) — a, b, c, R désignant les côtés d'un triangle ABC et le rayon du cercle circonscrit, trouver le maximum ou le minimum du rapport $(a^2+b^2+c^2) : R^2$ 51

15823. (J. J. Barniville, B.A., I.C.S.) — Having $u_n = u_{n-1} + u_{n-2}$, prove that

$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{4.7} + \frac{2}{5.11} + \frac{3}{7.16} + \dots = \frac{1}{2},$$

$$\frac{1}{1.2} - \frac{2}{2.4} + \frac{3}{4.5} - \frac{4}{5.7} + \frac{6}{7.11} - \dots = \frac{1}{3},$$

$$\frac{2}{1.5} - \frac{3}{2.7} + \frac{5}{4.11} - \frac{7}{5.16} + \frac{10}{7.23} - \dots = \frac{1}{4}$$ 68

15824. (D. Biddle.)—Give instances of quintets of integers in arithmetical progression, four being perfect squares placed in the following order :—(1) the first three and the last, (2) the last three and the first, (3) the first two and the last two, (4) all four together. Example of 1—7^2, 13^2, 17^2, 409, 23^2 107

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15825. (B. C. Wallis, F.C.P., B.Sc.Econ.) — What are the general formulæ to express the relationship between the lengths of the sides of a right-angled triangle in terms of one number n? *E.g.*, when the smallest side is an even number of units long the formula $(n+1)^2 = (n-1)^2 + 4n$ holds when n is a perfect square; also when the smallest side is an odd number of units long the formula $n^2 = (n-1)^2 + (2n-1)$ holds when n is such that $2n-1$ is a perfect square. These two formulæ enable us to write down a table showing the lengths of the sides when the short sides are the consecutive numbers beginning with 3..... 91

15826. (J. A. C., M.A.) — Solve, in integers, $x^2+y^2 = 5z^2$; and generally solve $x^2+y^2 = Az^2$, where A is not a square number 69

15827. (Robert W. D. Christie.) — Factorize 9,702,044,791, and show the principle is general for a certain class of numbers 90

15828. (*Communicated by* K. Deva Rao, B.A.) — A square board is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these squares be painted white, 4 black, 4 red, and 4 green, without repeating the same colour in the same vertical or horizontal row?..... 66

15830. (R. Chartres.)—A point P is taken within a square and joined to A, the extremity of the diagonal AC. Find the mean value (1) of the angle PAC, and (2) of its square..... 78

15832. (Professor Neuberg.) — Le centre A d'une sphère de rayon constant R parcourt une droite donnée d. Trouver (1) l'enveloppe du plan polaire π d'un point fixe P par rapport à la sphère mobile A, (2) la surface engendrée par la circonférence suivant laquelle le plan π coupe la sphère..... 71

15833. (Professor Cochez.)—Trouver le lieu des foyers d'une hyperbole dont on connait un sommet et une asymptote 76

15835. (The late R. Tucker, M.A.) — Given A, a, b in a triangle (the "ambiguous" case), if ω, ω' are the "B" angles of the triangle, prove

$$\cot\omega + \cot\omega' = 2b^2 \cot A/(cc')$$ 75

15837. (Professor Sanjána, M.A.)—Points P, Q, R are taken in the sides BC, CA, AB of a triangle such that the areas AQR, BRP, CPQ are each equal to a given area T. Show how to construct for the points, and prove that the distances of any two from the side not containing them are together equal to the corresponding altitude of the triangle 78, 112

15840. (R. Chartres.)—Find the mean distance of all points within a square from one of the corners in terms of a parabolic arc 95

15841. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Jack and Jill run a race of 300 yds. to the top of a hill, and then the same distance down the other side to a fixed goal. Going up the hill Jill takes 5 steps to Jack's 4, and going down takes 6 steps to Jack's 5. If Jack's speed in ascending is half what it is in descending, and if the length of Jill's step to Jack's is as 3 to 4, how much start should be given to Jill in order that they may reach the goal together?..... 104

15842. (C. Bickerdike.)—Three motorists start from three different towns and meet at a junction and, on comparing notes, find that the sum of the distances travelled was a minimum, the distance of the places from each (as the crow flies) being a, b, and c miles respectively. Find the speed of each car, all starting and meeting at the same time, the time from starting to meeting being n hours..... 85

15843. (Otto Meissner.)—Required to show, by a single trial, that 20857 is a prime..... 83

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15844. (Robt. W. D. Christie.) — Divide a straight line into two such parts that the square on one part shall equal three times the square on the other. Can it be solved generally? 99

15848. (R. F. Davis, M.A.)—Solve the equations

$$(x^3-3acx)^2 = 4a^3c^3, \quad a^3x^6+(b^3-3abc)x^3+c^3 = 0 \text{ ..... (i., ii.).}$$

..... 88

15849. (Professor Neuberg.)—On sait que les pieds M, M′, M″, M‴ pes normales abaissées d'un point P sur une conique Σ sont sur une hyperbole équilatère Σ′ passant par P et par le centre C de Σ. Étant donnés l'hyperbole équilatère Σ′ ($xy = 1$) et les trois points M, M′, M″ sur cette courbe, trouver (1) la relation entre les abscisses des points M, M′, M″, M‴; (2) la relation entre les abscisses des points M, M′, M″, P; (3) celle entre les abscisses des points M, M′, M″, C 95, 110

15852. (M. V. A. Sastry, B.A.)—The normals at the points P, Q, R, S of an ellipse pass through the same point O and cut the ellipse again at P′, Q′, R′, S′. If these points be concyclic, show that O lies on a hyperbola which cuts orthogonally all conics passing through the extremities of the axes of the ellipse 86

15854. (*The late* R. Tucker, M.A.)—ABC is a triangle; D, E, F are the points of contact of the ex-circles with the sides BC, CA, AB. Find (i.) area of DEF, (ii.) the Brocard angle of DEF. Prove also that the equation to the circle DEF is

$$2\Sigma a\beta\gamma = \Sigma aa\Sigma\left(\frac{a\cos^2\tfrac{1}{2}A-(b+c)\sin^2\tfrac{1}{2}A}{s-a}a\right) \text{ ..... } 89$$

15856. (Professor Coches.)—Trouver le lieu des points tels que leurs polaires relatives à trois cercles concourent en un même point 100

15862. (F. N. Mayers, M.A.)—Discuss the possibility or otherwise of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ 97

15864. (Robt. W. D. Christie.) — Prove that the Pellian equation $X_n^2-PY_n^2 = 1$ can be put into a trigonometrical form, and give the general solution. *E.g.*,

$$X_3 = \cos 3\theta = 4\cos^3\theta-3\cos\theta = 4x^3-3x,$$
$$Y_3 = \sin 3\theta = (4\cos^2\theta-1)\sin\theta = 4x^2y-y,$$

x and y having integral or fractional values..... 111

15871. (Professor Sanjána, M.A. Suggested by Question 15748.)—A, B, C are three fixed points on a conic, and from another fixed point K a chord is drawn terminated by the curve at P and Q. Find the locus of the intersection of AQ and BP, and investigate when it is of the first degree 97

ERRATA.

P. 55, l. 15. *For* "A^2" *read* "A."

P. 56, l. 18. *For* "u^{2n+1}" *read* "u_{2n+1}."

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

15621. (The late R. TUCKER, M.A.)—ABC is a triangle. Ω (or Ω') is a B-point, and ω_1, ω_2, ω_3 are the B-angles of the triangles ΩBC, ΩCA, ΩAB: prove that

$$\Sigma(\cot\omega_r) = \frac{1+\cot A\cot B\cot C\cot\omega+\cot^2\omega+\cot^4\omega}{\cot\omega-\cot A\cot B\cot C}.$$

Solution by the PROPOSER *and* K. S. PATRACHAN.

$$\cot\omega_1 = \cot\omega+\cot(C-\omega)-\cot C = \frac{(\cot\omega-\cot C)^2+\cot C\cot\omega+1}{\cot\omega-\cot C}$$

$$= \frac{1+\cot^2\omega}{\cot\omega-\cot C}-\cot C;$$

therefore

$$\Sigma\cot\omega_r = \frac{(1+\cot^2\omega)(3\cot^2\omega-2\cot^2\omega+1)}{\cot\omega-\Pi\cot A}-\cot\omega$$

$$= \frac{(1+\cot^2\omega)^2-\cot^2\omega+\Pi\cot A.\cot\omega}{\cot\omega-\Pi\cot A}$$

$$= \frac{1+\cot^2\omega+\cot^4\omega+\Pi\cot A.\cot\omega}{\cot\omega-\Pi\cot A}.$$

15751. (*Communicated by* J. A. H. JOHNSTON, M.A.)—Two light rods OA, OB, of equal length l, are hinged at a fixed point O, and carry heavy rings of equal mass m at the ends A and B. A third light smooth rod runs through the two rings and is hinged at a point P, vertically above O, at a distance l' from it ($l'>l$). The whole is started from the configuration in which A and B are vertically above and below O respectively, by impressing an angular velocity ω upon the rod PAB. Show that when the angle AOB between the rods is 2θ the reaction at P is

$$[2ml(\omega^2-g/l')(l'^4-l^4)\cos\theta]/[l'^2-l^2\cos 2\theta]^2.$$

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Solution by C. W. O'HARA.

At time t let C be the mid-point of AB, and let $\angle OPC = \phi$. Then, from the Question,

$$\angle COA = \angle COB = \theta; \quad OA = OB = l; \quad OP = l'.$$

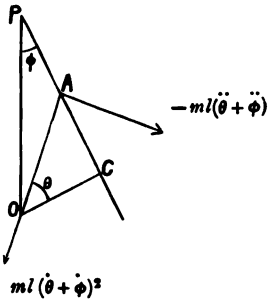
Also we obviously have the relation

$$l' \sin\phi = OC = l\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

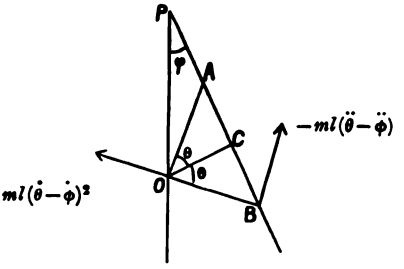
This determines ϕ in terms of θ.

Now, from the figure, the rod OA makes an angle $\frac{1}{2}\pi-(\theta+\phi)$ with OP, and the rod OB an angle $\frac{1}{2}\pi-(\theta-\phi)$ with PO produced. Hence the angular velocity and acceleration of ring A are $-(\dot\theta+\dot\phi)$ and $(\ddot\theta+\ddot\phi)$; and of ring B $-(\dot\theta-\dot\phi)$ and $-(\ddot\theta-\ddot\phi)$; whence the kinetic reactions for rings A and B are as shown in the diagrams :—

Ring A.

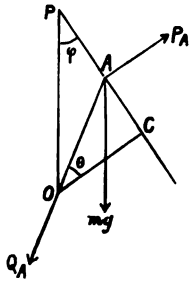


Ring B.

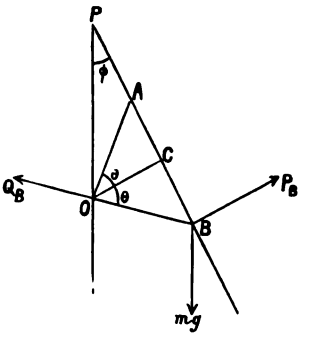


and the forces acting on the rings A and B are as shown in the diagrams :—

Ring A.



Ring B.



Hence, from the diagrams, resolving at right angles to OA, for ring A, we have

$$-ml(\ddot\theta+\ddot\phi) = P_A\sin\theta + mg\cos(\theta+\phi),$$

and, resolving at right angles to OB, for ring B, we have

$$-ml(\ddot\theta-\ddot\phi) = P_B\sin\theta - mg\cos(\theta-\phi);$$

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and therefore, by addition,

$$-2ml\ddot{\theta} = (P_A + P_B)\sin\theta + mg[\cos(\theta+\phi) - \cos(\theta-\phi)],$$

i.e.,

$$-2ml\ddot{\theta} = R\sin\theta - 2mg\sin\theta\sin\phi,$$

R evidently being the reaction we are required to determine.

Now $\ddot{\theta} = (d/d\theta)(\frac{1}{2}\dot{\theta}^2)$, and, from (1), $l'\sin\phi = l\cos\theta$. Hence the above equation becomes

$$R\sin\theta = -ml(d/d\theta)(\dot{\theta}^2) + 2mg(l/l')\cos\theta\sin\theta \quad \ldots\ldots\ldots\ldots(2).$$

To find $(d/d\theta)(\dot{\theta}^2)$, we first form the energy equation for the rings. At time t the velocities of the rings A, B are respectively $-(\dot{\theta}+\dot{\phi})l$ and $-(\dot{\theta}-\dot{\phi})l$, and therefore their kinetic energy is

$$\tfrac{1}{2}m[l^2(\dot{\theta}+\dot{\phi})^2 + l^2(\dot{\theta}-\dot{\phi})^2],$$

i.e., $ml^2(\dot{\theta}^2+\dot{\phi}^2)$. Now, by differentiating, with respect to t,

$$l'\sin\phi = l\cos\theta,$$

we get

$$l'\cos\phi\dot{\phi} = -l\sin\theta\dot{\theta}\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence we find $\dot{\phi}^2 = [l^2\sin^2\theta/(l'^2 - l^2\cos^2\theta)]\dot{\theta}^2$.

Hence kinetic energy

$$= ml^2\dot{\theta}^2[1 + l^2\sin^2\theta/(l'^2 - l^2\cos^2\theta)] = ml^2\dot{\theta}^2[(l'^2 - l^2\cos 2\theta)/(l'^2 - l^2\cos^2\theta)].$$

Initially, $\dot{\phi} = \omega$, $\phi = 0$, $\theta = \frac{1}{2}\pi$. Hence, from (3), $l'\omega = -l\dot{\theta}_0$, where $\dot{\theta}_0$ is the initial value of $\dot{\theta}$, and therefore the initial kinetic energy is $m\omega^2(l^2+l'^2)$. Also the height of C, the centroid of the rings A, B above O, is $l'(1-\cos^2\phi) = l'\sin^2\phi = (l^2/l')\cos^2\theta$, and therefore the potential energy of the system, the initial configuration being standard configuration, is $2mg(l^2/l')\cos^2\theta$. Hence the energy equation is

$$m\omega^2(l'^2+l^2) - ml^2\dot{\theta}^2[(l'^2 - l^2\cos 2\theta)/(l'^2 - l^2\cos^2\theta)] = 2mg(l^2/l')\cos^2\theta.$$

This equation determines $\dot{\theta}^2$ in terms of θ, and, differentiating with respect to θ, we easily find $(d/d\theta)(\dot{\theta}^2)$. Substituting the value of $(d/d\theta)(\dot{\theta}^2)$ thus found in (2), we can find R in terms of θ. The result is

$$R = 2ml\cos\theta\{[(l'^4 - l^4)(\omega^2 - g/l')]/(l'^2 - l^2\cos 2\theta)^2\}.$$

Cubic Equations: a direct method of solving them (when irreducible by Cardan's method) to any required degree of approximation.

By D. Biddle.

Reduce the equation to the form

$$x^3 + qx + r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha).$$

Let

$$x = (r/q)u \quad \text{and} \quad q^3/r^2 = -c \quad \ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

Then we obtain the following equalization of known quantities:—

$$u^3 - cu - c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(α) belongs to a group, infinite in extent, of which the standard is of form

$$X^3 + QX + R = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and each individual equation resembles

$$x^3 + Qn^2x + Rn^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

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where m may be either integral or fractional and $x = mX$. For the whole group under the particular standard, the quantities in (1), both known and unknown, are identical. Therefore when (1) is solved for any particular standard the whole group belonging to that standard is practically solved also: adjustment of values, according to (β), is then alone necessary. Thus, if we are able to solve $X^3-7X-6=0$, it is easy to solve $x^3-28x-48=0$, because $28 = 7 \cdot 2^2$ and $48 = 6 \cdot 2^3$, whence we have $x = 2X$. Now, by (1),

$$c = u^3/(u+1) = u^2-u+1-\Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4, 5),$$

where $\Delta = 1/(u+1)$. By (5),

$$u = \tfrac{1}{2}[1 \pm (4c-3+4\Delta)^{\frac{1}{2}}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

For our present purpose, the sign preceding the quantity under the radical sign may be regarded as always positive, because it is only in cases in which c exceeds $6\frac{3}{4}$ that Cardan's method is inoperative. At that value of c, the values of u are 3, $-\frac{3}{2}$, $-\frac{1}{2}$; the positive value of u increases with c, and the positive value of u is the only one we need seek, as this will enable us to find the corresponding value of x, the two remaining values of which can then easily be found from the quadratic equation to which the cubic (α) becomes reducible.

Taking (6) as our formula, we can carry out the following process, until the required degree of approximation is reached. It is neither laborious nor prolonged, and, though expedited by the use of logarithms and reciprocals, is in reality independent of all tables.

$$\text{Let} \quad u_1 = \tfrac{1}{2}[1+(4c-3)^{\frac{1}{2}}], \quad u_2 = \tfrac{1}{2}\{1+[4c-3+4/(u_1+1)]^{\frac{1}{2}}\} \ldots(7, 8),$$

$$u_3 = \tfrac{1}{2}\{1+[4c-3+4/(u_2+1)]^{\frac{1}{2}}\} \ldots\ldots\ldots\ldots\ldots(9),$$

and so on. The convergence is very rapid, but some oscillation, by way of alternate increase and decrease in results, occurs.

Example.—Solve $x^3-17x-20=0$, giving results correct (say) to five places of decimals. Here $c = 17^3/20^2 = 12\frac{1}{4}\frac{3}{0}\frac{3}{0}$; and the stages are as follows:—

n	$4c-3+4/(u_{n-1}+1)$	$[4c-3+4/(u_{n-1}+1)]^{\frac{1}{2}}$	u_n	$4/(u_n+1)$
1	46·130000	6·791906	3·895953	0·817012
2	46·947012	6·851789	3·925895	0·812050
3	46·942050	6·851426	3·925713	0·812065
4	46·942065	6·851428	3·925714	

Therefore we are justified in regarding $u = 3{\cdot}9257135$ as its positive value; and, corresponding to this, $x = \frac{1}{1}\frac{2}{0}u = 4{\cdot}618486$. For the two remaining roots of x, we are able to form the following quadratic equation, by reduction of the cubic:—$x^2+4{\cdot}618486x+4{\cdot}330423=0$, whence we obtain $x+2{\cdot}309243 = \pm 1{\cdot}001090$. Thus, the three roots of x are 4·618486, −1·308153, −3·310333. Their products taken two at a time sum to −16·999995 instead of −17, and the product of all three is defined by the logarithm 1·3010309, instead of 1·3010300, which is the logarithm of 20.

N.B.—It is sometimes possible to see that the convergence is towards an exact value, on assuming which an absolute result is obtained.

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15713. (Professor LAUVERNAY.)—Dans un triangle rectangle ABC, α, β, γ sont les points de contact de cercles ex-inscrits respectivement avec les côtés BC, CA, AB. Démontrer que, si les droites Aα, Bβ, Cγ sont concourantes sur la circonférence inscrite à ce triangle, les côtés de ce triangle sont proportionnels aux nombres 3, 4, 5.

Solutions (I.) *by* Prof. SANJÁNA, M.A.; (II.) *by the late* R. TUCKER, M.A. (III.) *by* C. M. Ross.

(I.) The equations of the lines Aα, Bβ, Cγ are

$$\beta/\gamma = (s-b)\sin C/[(s-c)\sin B],\ \ldots,\ \ldots;$$

hence their point of concurrence is

$$\beta b/(s-b) = \gamma c/(s-c) = \alpha a/(s-a).$$

For this to be on the in-circle, we must have $\Sigma \cos A \sqrt{\alpha} = 0$; thus

$$\sqrt{[(s-a)/a]}\sqrt{\{[s(s-a)]/bc\}} \pm \sqrt{[(s-b)/b]}\sqrt{\{[s(s-b)]/ca\}} \pm \sqrt{[(s-c)/c]}\sqrt{\{[s(s-c)]/ab\}} = 0;$$

so that $(s-a)\pm(s-b)\pm(s-c) = 0$. Taking $s-a+s-b-(s-c) = 0$, we get $3c = a+b$. If A be the right angle, we have also $a^2 = b^2+c^2$; whence we obtain $a = \frac{4}{5}c$, $b = \frac{3}{5}c$, *i.e.*, $a : b : c = 5 : 4 : 3$.

(II.) The in-circle is

$$x^2+y^2-2(s-a)(x+y)+(s-a)^2 = 0 \quad \ldots\ldots\text{(i.)}.$$

The line BC is $bx+cy-bc = 0$ $\ldots\ldots\text{(ii.)}$,

and the intersection (P) of Bβ, Cγ is

$$x/c+y/(s-c) = 1, \quad x/(s-b)+y/b = 1,$$

i.e., $$x = [c(s-b)]/s, \quad y = [b(s-c)]/s \quad \ldots\ldots\text{(iii.)}.$$

Now (iii.) is on (i.); therefore

$$c^2(s-b)^2+b^2(s-c)^2-2(s-a)s[(b+c)s-2bc]+s^2(s-a)^2 = 0 \ \ldots\text{(iv.)}.$$

Also, equating two expressions for the radius, we get

$$bc = 2s(s-a) \quad \ldots\ldots\text{(v.)}.$$

Hence (iv.) gives $4a^2s^2-12bc(b+c)s+17b^2c^2 = 0$,

i.e., $$a^2(a+b+c)^2-6abc(a+b+c)+5b^2c^2 = 0,$$

i.e., $$[a(a+b+c)-5bc][a(a+b+c)-bc] = 0.$$

Hence $2sa = 5bc = 10s(s-a)$ by (v.); therefore $a = 5(s-a)$ or $6a = 5s$, and this, with $b^2+c^2 = a^2$, gives $a : b : c = 5 : 4 : 3$.

(III.) It is known that C$\alpha = s-b$ and C$\beta = s-a$. Hence the equations of Aα, Bβ, Cγ are $x/(s-b)+y/b = 1$, $x/a+y/(s-a) = 1$, and $x/[a(s-b)] = y/[b(s-a)]$ (axes AC and CB). These lines intersect in the point $\delta[a(s-b)/s,\ b(s-a)/s]$. It is easily proved that Cγ passes through this point since the determinant

$$\begin{vmatrix} b, & s-b, & b(s-b) \\ s-a, & a, & a(s-a) \\ b(s-a), & a(s-b), & 0 \end{vmatrix}$$

vanishes.

Again the equation of the in-circle is

$$(x-s+c)^2+(y-s+c)^2 = (s-c)^2 \quad \ldots\ldots\text{(1)}.$$

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Also the co-ordinates of the point δ may be written

$$[\sqrt{(a^2+b^2)}-b,\ \sqrt{(a^2+b^2)}-a],$$

since $c^2 = a^2+b^2$; therefore, substituting these values for x and y in (1), it becomes $13a^2+5ab+13b^2-11(a+b)(a^2+b^2)^{\frac{1}{2}}=0$. Squaring and rearranging, $48(a^2+b^2)^2-112ab(a^2+b^2)+25a^2b^2=0$, *i.e.*,

$$[12(a^2+b^2)-25ab][4(a^2+b^2)-ab]=0.$$

From this equation $4a-3b=0$; therefore $a:b=3:4$; therefore the lines $A\alpha$, $B\beta$, $C\gamma$ will intersect on the in-circle if $a:b:c=3:4:5$.

15495. (The late Rev. Prebendary WHITWORTH, M.A.)—Show geometrically that, if ABCD be a quadrilateral inscribed in a circle, and if AP, AQ, AR be perpendiculars from A on BC, CD, BD respectively, then

$$BC/AP + CD/AQ = BD/AR.$$

[It is impossible to publish more than a few typical solutions out of the large number sent in.—ED.]

Solutions (I.) *by* J. H. CARSON, B.A., Rev. T. WIGGINS, B.A., *and others*; (II.) *by* W. SCRIMGEOUR, M.A., B.Sc., *and* S. C. BAGCHI, B.A.; (III.) *by* C. M. ROSS; (IV.) *by* T. B. RAMACHANDRAN, B.A., *and others*.

(I.) P, Q, R are collinear (Simson's line). The triangles ABR and ACQ are similar; therefore

$$BR/RA = CQ/AQ \quad \ldots\ldots(1).$$

The triangles ARD and APC are similar; therefore

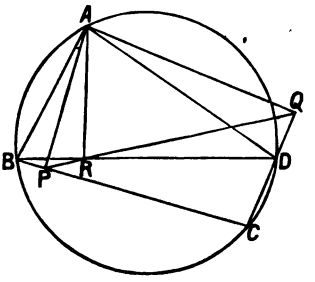
$$RD/RA = PC/AP \quad \ldots\ldots(2).$$

The triangles BAP and DAQ are similar; therefore

$$BP/AP = DQ/AQ \quad \ldots\ldots(3).$$

Therefore, adding (1) and (2), and subtracting (3),

$$BD/AR = BC/AP + CD/AQ.$$



(II.) [For figure see Solution (I.).] RQ is the Simson line of the point A with respect to the triangle BCD; RAQD, PBAR, PCQA are cyclic quadrilaterals.

$$\begin{aligned} BC.AQ.AR + CD.AP.AR &= \frac{BC.2\triangle QAR}{\sin QAR} + \frac{CD.2\triangle PAR}{\sin PAR} \\ &= \frac{BC.2\triangle QAR}{\sin BDC} + \frac{CD.2\triangle PAR}{\sin DBC} \\ &= \frac{BD\,(2\triangle QAR + 2\triangle PAR)}{\sin BCD} \\ &= \frac{BD}{\sin BCD}(2\triangle PAQ) = \frac{BD.2\triangle PAQ}{\sin PAQ}; \end{aligned}$$

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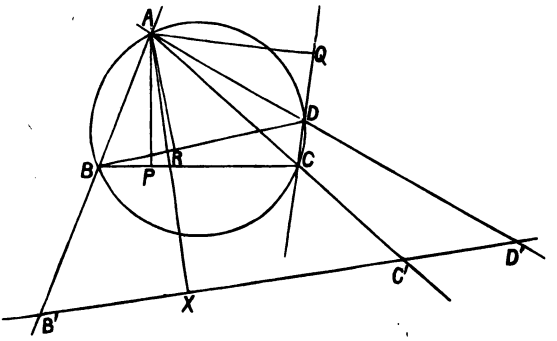
therefore $BC.AQ.AR + CD.AP.AR = BD.AP.AQ$;
therefore $BC/AP + CD/AQ = BD/AR$
[since $\angle QAR = \angle BDC$, $\angle PAR = \angle DBC$].

(III.) Let AB, AC, AD produced cut a line B′C′D′ lying in the plane of the circle in the points B′, C′, D′. Draw AX perpendicular to B′C′D′; then

$$B'C'/AX + C'D'/AX = B'D'/AX \quad \text{.....(A).}$$



Let this expression be inverted from A; then, if B, C, D are the inverses of B′, C′, D′, $AB'.AB = AC'.AC$, ... Therefore

$$AB' : AC' = AC : AB;$$

therefore the triangles B′AC′, BAC are equiangular (Euc. VI. 6); therefore $B'C' : BC = AX : AP$. Similarly

$$C'D' : CD = AX : AQ, \quad B'D' : BD = AX : AR.$$

Hence, from the relation (A), $BC/AP + CD/AQ = BD/AR$.

(IV.) Join AC. Then, from $\triangle ADC$,

$DA.AC = AQ.\phi$ (Euc. VI. c.)...(1);

from $\triangle ADB$,

$DA.AB = AR.\phi$ .....(2);

from $\triangle ACB$,

$CA.AB = AP.\phi$ .....(3),

where ϕ = length of diameter of circle ABCD. Also

$BD.AC = AB.CD + BC.AD$ (Euc. VI. D);

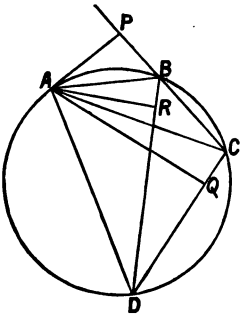
therefore

$(BD.AC)/AR$
$= AC(AB.CD + BC.AD)/(AB\ AQ)$

from (1) and (2), *i.e.*,

$BD/AR = CD/AQ + (BC.AD)/(AQ.AB)$,

i.e., $BD/AR = CD/AQ + BC/AP$, from (3) and (1).



15753. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Solve in integers

$$\tfrac{1}{2}(q^2 + 1) = p^4.$$

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Solutions (I.) *by* R. W. D. CHRISTIE, R. F. DAVIS, M.A., *and* A. H. BELL ; (II.) *by the* PROPOSER.

(I.) We have infinite solutions of the equation $2q^2-p^2=1$. If any one of them $q=t^2$, then $2t^4-p^2=1$, and the equation required is solved. Let $t=2y^2+2yz+z^2$. We have generally

$$2(2y^2+2yz+z^2)^4=(4y^4+16y^3z+12y^2z^2-z^4)^2+(4y^4-12y^2z^2-8yz^3-z^4)^2 \quad \ldots\ldots\ldots\ldots(1).$$

If $y=1$ and $z=1$, this becomes

$$2(2y^2+2y+1)^4=(4y^4+16y^3+12y^2-1)^2+(4y^4-12y^2-8y-1)^2 \ldots(2),$$

$$2(2+2z+z^2)^4=(4+16z+12z^2-z^4)^2+(4-12z^2-8z^3-z^4)^2 \ldots\ldots\ldots(3).$$

Now, if we can make $4y^4-12y^2-8y-1=\pm1$, the problem is solved; *e.g.*, $4y^4=12y^2+8y$ or $y^3=3y+2$ $(y=2)$ and $2(13)^4=239^2+(-1)^2$ by (2), and similarly with the equation $4-12z^2-8z^3-z^4=\pm1$, or indeed any of the six dexter squares. The solution involves that of a biquadratic, but probably the work of securing integers would be laborious. The theorem is deeply interesting and in this connexion one may use the oscillating equation of the convergents given in Vol. III.

(II.) Here $q^2-2(p^2)^2=-1$, and the solutions of $\tau-2.v^2=-1$ must be picked out which have $v=\square$. The only solution (until $q>1{,}600{,}000$) is $239^2-2.13^4=-1$.

[For a table of such solutions, see the PROPOSER's new *Tables of Quadratic Partitions*, p. 261.

15721. (Professor M.W. CROFTON, F.R.S.)—Two players A., B., whose chances of winning a game are p, q $(p+q=1)$, put down each a stake, and play on the condition that whoever wins two games running shall receive the whole. Show that in fairness their stakes ought to be in the proportion $p^2(1+q):q^2(1+p)$.

Solution by F. N. MAYERS.

The game will terminate at the $2n$-th game unless there is an alternation of wins by each player in succession, the probability of which is $2p^nq^n$, or p^nq^n if the alternation is to begin with a win by an assigned player.

If A. is to win precisely at the $2n$-th game, there must be an alternation for the previous $2n-2$ games in which A. loses the last game, followed by two wins by A.; the probability of this is $p^{n-1}q^{n-1}\times p^2=p^{n+1}q^{n-1}$. If A. is to win at the $(2n-1)$-th game precisely, there must be alternation in the previous $2n-2$ games, in which A. must win the last game, followed by a win by A.; the probability of this is $p^{n-1}q^{n-1}\times p=p^nq^{n-1}$. The probability that he wins at or before the $2n$-th game is the sum of the probabilities that he wins at the $2n$-th, $(2n-1)$-th, ..., 2nd game precisely.

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Hence we get for the required probability the sums of the two following series:—

$$p^{n+1}q^{n-1} + p^{n-1}q^{n-1} + \ldots + p^2 = p^2[1 + \ldots + (pq)^{n-1})] \quad \ldots\ldots\ldots(1),$$

and $p^n q^{n-1} + \ldots + p^2 q$ (the first game being excluded)

$$= p^2 q[1 + pq + \ldots + (pq)^{n-2}] \ldots (2).$$

The sum of (1) is $p^2\{[1-(pq)^n]/(1-pq)\}$, and that of (2) is

$$p^2 q\{[1-(pq)^{n-1}]/(1-pq)\}.$$

Making n infinite, the sum of (1) and (2) is $[p^2(1+q)]/(1-pq)$. Similarly probability of B. winning is $[q^2(1+p)]/(1-pq)$; hence the required result.

If we make the number of games *odd* instead of *even*, *i.e.*, $2n+1$ instead of $2n$, it will only add the infinitesimal term $p^{n+1}q^n$ to series (2).

15677. (Robt. W. D. Christie.)—Let p be any prime and the letters all *different* integers. Prove that

$$pqr = l^3 + l^2 s + lt + u = (\mathrm{LMN})^{\frac{1}{3}}, \ \ldots;$$

e.g., $103.18.35 = 25^3 + 25^2.53 + 25.635 + 265 = (630.3605.1854)^{\frac{1}{3}}$.

[N.B.—The original form of the Question has been slightly modified by the Proposer so as to ensure clearness —Ed.]

Solution by the Proposer.

$$pqr = l^3 + l^2 s + lt + u = (\mathrm{LMN})^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{A}).$$

This theorem depends on the two fundamental theorems

$$r^{2m} + 1 = 0 \bmod p \quad \text{and} \quad ab - (a+b)[ab-r]/(a+b) = r,$$

a root, where in the case of $p = 4m+1$ we may put $r = 1$ always. Thus theorem (A) in this case simplifies considerably. Let

$l = (ab-r)/p$ $= 25$ in the given case, $\mathrm{L} = l^2 + r = 630$,
$s = (a^2 + b^2 + 2r)/p$ $= 53$,, ,, $\mathrm{M} = a^2 + r = 3605$,
$t = qr + \mathrm{R}$ $= 635$,, ,, $\mathrm{N} = b^2 + r = 1854$,
$u = \mathrm{R}[a^2 + b^2 + 2\mathrm{R}]/(a+b) = 265$,, ,,
q = residue of $[(r^{m+1})^2 + r]/p$ and r = any root of p,
r = residue of $[(r^{3m+2})^2 + r]/p$.

In the example given take $r = 5$, the smallest root of 103. Then

$$\left.\begin{array}{l} a = \text{residue of } 5^{26} \bmod p = 60 \\ b = \text{residue of } 5^{77} \bmod p = 43 \end{array}\right\} = 103.$$

Then the equation $ab-(a+b)[(ab-r)/(a+b)] = r$ becomes

$$60.43 - 103.25 = 5;$$

thus $l = 25$. Again $q = (60^2+5)/p = 35$, $r = (43^2+5)/p = 18$. Consequently in the given equation we have

$$103.18.35 = 25^3 + 25^2.53 + 25.635 + 265 = (630.3605.1854)^{\frac{1}{3}}.$$

Similarly for root 6 the figures would be

$$103.10.49 = 22^3 + 22^2.59 + 22^1.496 + 354 = (490.1030.5047)^{\frac{1}{3}},$$

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and we may have as many forms as there are roots, *i.e.*, 32 sets for the prime 103. In this connexion the theorems by the Solver in former volumes may be consulted with advantage.

The theorems may also be written

$$(A)\quad pqr = \left[\left(\frac{ab-r}{a+b}\right)^3 + \left(\frac{ab-r}{a+b}\right)^2\left(\frac{a^2+b^2+2r}{a+b}\right) + \left(\frac{ab-r}{a+b}\right)(qr+R)\right] + \left[\left(\frac{a^2+b^2+2r}{a+b}\right)(R)\right] = (LMN)^{\frac{1}{3}}\ldots$$

for primes of form $4m-1$, and

$$(B)\quad pqr = \left[\left(\frac{ab-1}{a+b}\right)^3 + \left(\frac{ab-1}{a+b}\right)^2\left(\frac{a^2+b^2+2}{a+b}\right) + \left(\frac{ab-1}{a+b}\right)(qr+1)\right] + \left[\left(\frac{a^2+b^2+2}{a+b}\right)\right] = (LMN)^{\frac{1}{3}}\ldots,$$

for primes of form $(4m+1)$.

The theorem leads indirectly to one or two other theorems, one of which is $a^2+b^2+a^2+b^2+(ab+1)^2 = (a+b)^2+1^2+a^2+b^2+(ab)^2$,

where it will be noticed, the sum of the roots being equal, we may add n to the root of each square, &c.

10790. (*Corrected.* H. W. Segar, B.A.)—Show that

$$\tfrac{1}{4}\pi\int_0^{\frac{1}{2}\pi}\log(1+\cos\theta\cos x)\,\frac{dx}{\cos x} = \frac{\cos\theta}{1^3} - \frac{\cos 3\theta}{3^3} + \frac{\cos 5\theta}{5^3} - \ldots.$$

Solution by Professor Sanjána, M.A.

The left side is known to be $\frac{1}{4}\pi\frac{1}{2}(\frac{1}{4}\pi^2-\theta^2) = \frac{1}{32}\pi^3-\frac{1}{8}\pi\theta^2$. Calling the right side $\phi(\theta)$, we have $\phi'(\theta) = -\sin\theta/1^2+\sin 3\theta/3^2-\sin 5\theta/5^2+\ldots$. $\phi''(\theta) = -\cos\theta/1+\cos 3\theta/3-\cos 5\theta/5+\ldots$. Let $2\cos\theta = x+x^{-1}$; then

$$\begin{aligned}2\phi''(\theta) &= -x/1+x^3/3-x^5/5\ldots-x^{-1}/1+x^{-3}/3-x^{-5}/5+\ldots\\ &= -(\tan^{-1}x)-(\tan^{-1}1/x)\ \text{(by Gregory's series)}\\ &= -\tan^{-1}\infty = -\tfrac{1}{2}\pi.\end{aligned}$$

Thus, $\phi''(\theta) = -\frac{1}{4}\pi$. Integrating, $\phi'(\theta) = A-\frac{1}{4}\pi\theta$; and putting $\theta = 0$, we get $A = 0$. Thus $\phi'(\theta) = -\frac{1}{4}\pi\theta$; hence

$$\phi(\theta) = B-\tfrac{1}{8}\pi\theta^2.$$

Putting $\theta = 0$, we get $B = 1/1^3-1/3^3+1/5^3-\ldots = \pi^3/2^5$. Thus

$$\phi(\theta) = \tfrac{1}{32}\pi^3-\tfrac{1}{8}\pi\theta^2;$$

hence the result.

15748. (A. M. Nesbitt, M.A.)—A, B, K are three fixed points on a tangent to a conic, and the other tangents from A and B meet in C. Any straight line through K cuts the conic in P and Q, and AQ, BP meet in R. Prove that the locus of R is a conic inscribed in the triangle ABC.

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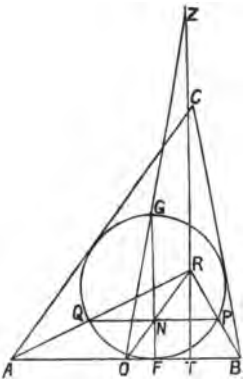
Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* J. A. H. Johnston, M.A., *and* Professor Sanjàna, M.A.



(I.) This theorem is the generalization by conical projection of the following particular case:—If PQ be a variable chord of the inscribed circle of the triangle ABC drawn parallel to AB, then the locus of the point of intersection R of AQ, BP is a conic inscribed in ABC.

Let O be the middle point of AB and FG the diameter of the inscribed circle through the point of contact F with AB. Draw RT perpendicular to AB and let it meet OG produced in Z. Then

$$ZR.RT : GN.NF = OR^2 : ON^2$$

and $PN^2 (= GN.NF) : OB^2 = RN^2 : OR^2$; therefore

$$ZR.RT : OB^2 = RN^2 : ON^2 = TF^2 : OF^2,$$

or ZR.RT varies as TF^2. Hence the locus of R is a conic having OF, OG for tangents and FG for the corresponding chord of contact. It is, moreover, evident by an examination of the figure that, since P, Q always lie within ABC, the locus of R reaches both CA, CB, but does not cross these lines. Therefore CA, CB are tangents to the locus.

(II.) Let ABC be the triangle of reference, and the equation of the conic $\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0$. Ignoring ambiguities of sign throughout, let us call P the point x_1, y_1, z_1 and Q x_2, y_2, z_2. The equation to PQ is

$$x\sqrt{l}\,[\sqrt{(y_1z_2)} + \sqrt{(y_2z_1)}] + y\sqrt{m}\,[\sqrt{(x_1z_2)} + \sqrt{(x_2z_1)}] + z\sqrt{n}\,[\sqrt{(x_1y_2)} + \sqrt{(x_2y_1)}] = 0.$$

If it pass through the point K $(k, k', 0)$, then

$$k\sqrt{l}\,[\sqrt{(y_1z_2)} + \sqrt{(y_2z_1)}] + k'\sqrt{m}\,[\sqrt{(x_1z_2)} + \sqrt{(x_2z_1)}] = 0 \quad \text{(i.)}$$

where
$$\sqrt{(l/n)}\,\sqrt{(x_1/z_1)} + \sqrt{(m/n)}\,\sqrt{(y_1/z_1)} + 1 = 0$$
and
$$\sqrt{(l/n)}\,\sqrt{(x_2/z_2)} + \sqrt{(m/n)}\,\sqrt{(y_2/z_2)} + 1 = 0 \quad \text{(ii.)}.$$

Also R, (x, y, z) lies on AQ and BP; therefore

$$y/z = y_2/z_2, \quad x/z = x_1/z_1 \quad \text{(iii.)}.$$

Divide (i.) throughout by $\sqrt{(z_1z_2)}$ and substitute in it the values of x_1/z_1, x_2/z_2, y_1/z_1, y_2/z_2 derived from (iii.) and (ii.), and the locus appears as

$$k\sqrt{l}\,[1 - \sqrt{(l/n)}\,\sqrt{(x/z)} + \sqrt{(m/n)}\,\sqrt{(y/z)}] + k'\sqrt{m}\,[\sqrt{(l/n)}\,\sqrt{(x/z)} + 1 - \sqrt{(m/n)}\,\sqrt{(y/z)}] = 0,$$

or, if we multiply this result by $\sqrt{(nz)}$,

$$\sqrt{(lx)} - \sqrt{(my)} - [(k\sqrt{l} + k'\sqrt{m})/(k\sqrt{l} - k'\sqrt{m})]\sqrt{(nz)} = 0,$$

which is a conic inscribed in ABC.

Mr. W. F. Beard, M.A., also solves the problem by the theory of projection.

Professor Sanjàna remarks that the reciprocal theorem may be enun-

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ciated thus:—A conic circumscribes the triangle ABC, and CK is a fixed straight line. From a varying point K in CK tangents are drawn to the conic, meeting CA, CB severally in M, N. Prove that the envelope of MN is a circum-conic of ABC.

15761. (Professor Genese.)—Prove that there always exists one point which has the same tetrahedral co-ordinates with respect to any two given tetrahedra.

Solution by J. A. H. Johnston, M.A.

Let s_1, s_2, s_3, s_4 and s'_1, s'_2, s'_3, s'_4 be the faces of the two tetrahedra and V, V′ their volumes. Then, if p and p' be perpendiculars from a point on s_1 and s'_1, the x co-ordinates of this point are $\frac{1}{3}ps_1/\mathrm{V}$ and $\frac{1}{3}p's'_1/\mathrm{V}'$ with reference to the two tetrahedra. If these be the same, $p/p' = \mathrm{V}s'_1/\mathrm{V}'s_1$, and this restricts the position of the point to a determinate plane through the intersection of s_1 and s'_1. If the x, y, and z co-ordinates are the same, the point is therefore determined as the point intersection of three definite planes. Finally, since $x+y+z+u = 1$ for both systems, the four co-ordinates are the same for this point of intersection.

15478. (Professor Nanson.)—If r, θ are the focal polar co-ordinates of the foot of a normal from r', θ' to a conic, prove that

$$r_1r_2r_3r_4 = b^2e^{-2}r'^2, \quad \theta_1+\theta_2+\theta_3+\theta_4 = 2\theta'+(2n+1)\pi.$$

Solution by the Proposer.

Using circular co-ordinates ξ, η with origin at a focus, any conic is $2\xi\eta = \varpi^2$, where $\varpi = a\xi+b\eta+c$, and the normal at (ξ, η) is

$$(\xi'-\xi)/(\xi-b\varpi) = (\eta'-\eta)/(\eta-a\varpi).$$

Eliminating ϖ, η from these three equations, we get for ξ a quartic whose roots satisfy $\xi_1\xi_2\xi_3\xi_4 = -k^2\xi'^2$, where k depends only on a, b, c. Now $\xi = r\cos\theta$, whence we get

$$\theta_1+\theta_2+\theta_3+\theta_4 = 2\theta'+(2n+1)\pi, \quad r_1r_2r_3r_4 = k^2r'^2,$$

and, taking the normals from the centre, we find $b = ke$. For another solution see *Reprint*, Vol. vi. (New Series), p. 94.

15766. (A. M. Nesbitt, M.A.)—The locus of the intersection of normals at the extremities of conjugate diameters of $x^2/a^2+y^2/b^2 = 1$ is the curve $2(b^2y^2+a^2x^2)^3 = (a^2-b^2)^2(a^2x^2-b^2y^2)^2$.

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Solutions (I.) *by* J. A. H. JOHNSTON, M.A.; (II.) *by* M. V. A. SASTRY, B.A., C. M. ROSS, *and others*.

(I.) The locus point is common to

$$ax\lambda - by\mu - (a^2 - b^2) = 0, \quad by\lambda + ax\mu + (a^2 - b^2) = 0,$$

where $\lambda = 1/\cos\phi$, $\mu = 1/\sin\phi$, or $1/\lambda^2 + 1/\mu^2 = 1$;

therefore

$\lambda = [(a^2 - b^2)(ax - by)]/(a^2x^2 + b^2y^2)$, $\mu = [-(a^2 - b^2)(ax + by)]/(a^2x^2 + b^2y^2)$,

and $1/\lambda^2 + 1/\mu^2 = 1$ gives

$$(a^2x^2 + b^2y^2)^2/[(a^2 - b^2)^2(ax - by)^2] + (a^2x^2 + b^2y^2)^2/[(a^2 - b^2)^2(ax + by)^2] = 1$$

or $$(a^2x^2 + b^2y^2)^2[(ax - by)^2 + (ax + by)^2] = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2;$$

therefore $$2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$$

(II.) Let θ and $\theta + \frac{1}{2}\pi$ or $\theta + \frac{3}{2}\pi$ be the eccentric angles of the extremities of a pair of conjugate diameters. The equations of the normals are

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta \quad \ldots\ldots\ldots\ldots (1),$$

$$ax\cos\theta + by\sin\theta = \mp(a^2 - b^2)\sin\theta\cos\theta \quad \ldots\ldots\ldots\ldots (2).$$

Squaring (1) and (2) and adding, we have

$$a^2x^2 + b^2y^2 = \tfrac{1}{4}(a^2 - b^2)^2\sin^2 2\theta \quad \ldots\ldots\ldots\ldots (3).$$

Squaring (1) and (2) and subtracting, we have

$$(a^2x^2 - b^2y^2)\cos 2\theta + 2abxy\sin 2\theta = 0;$$

therefore $$(a^2x^2 - b^2y^2)^2\cos^2 2\theta = 4a^2b^2x^2y^2\sin^2 2\theta;$$

therefore $(a^2x^2 - b^2y^2)^2 = (a^2x^2 + b^2y^2)^2\sin^2 2\theta = 2(a^2x^2 + b^2y^2)^3/(a^2 - b^2)^2$, by (3); therefore the equation of the locus is

$$2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$$

15690. (The late R. TUCKER, M.A.)—ABC is a triangle, K is the S. point, and ω_1, ω_2, ω_3 are the B. angles of the triangles BKC, CKA, AKB. Prove that $\Sigma\cot\omega_r = 6\cot\omega - (\operatorname{cosec}^2\omega + 2)/(2\cot\omega)$.

Solutions (I.) *by the* PROPOSER; (II.) *by* KELA DEVA RAU, B.A., *and* P. V. SESHU.

(I.) $\sin(B - \theta)/\sin\theta = c/a$;

therefore $\cot\theta = (c + a\cos B)/(a\sin B)$,

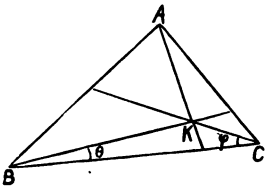
and similarly

$$\cot\phi = (b + a\cos C)/(a\sin C).$$

Now $\cot\omega_1 = \cot\theta + \cot\phi - \cot(\theta + \phi)$,

and $\cot\theta + \cot\phi = \Sigma a\sin A/(a\sin B\sin C)$

$= k/2\Delta = 2\cot\omega$,



and $\cot\theta\cot\phi - 1$

$= [bc + a(b\cos B + c\cos C) + a^2\cos B\cos C - a^2\sin B\sin C]/(bc\sin^2 A)$

$= \operatorname{cosec}^2 A + 2\cot B\cot C$;

therefore $$\Sigma\cot\omega_r = 6\cot\omega - (\operatorname{cosec}^2\omega + 2)/2\cot\omega.$$

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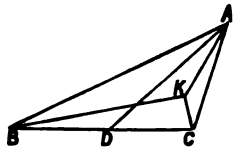
(II.) Bisect BC at D. Join AD.



$$\frac{AB}{BD} = \frac{\sin ADB}{\sin BAD} = \frac{\sin (B + BAD)}{\sin BAD},$$

i.e., $$\frac{2 \sin (A + B)}{\sin A} = \frac{\sin (B + BAD)}{\sin BAD};$$

therefore $2 (\cot A + \cot B) = \cot B + \cot BAD;$

therefore $$\cot BAD = 2 \cot A + \cot B.$$

Now $$\cot \omega_2 = \cot KAC + \cot ACK + \cot CKA.$$

But $$\cot KAC = \cot BAD = 2 \cot A + \cot B.$$

Similarly $$\cot ACK = 2 \cot C + \cot B,$$

$$\cot CKA = -\cot (KAC + ACK) = \frac{1 - \cot KAC \cot ACK}{\cot KAC + \cot ACK}$$

$$= \frac{1 - (2 \cot A + \cot B)(2 \cot C + \cot B)}{2 \cot A + \cot B + 2 \cot C + \cot B}$$

$$= \frac{1 - \cot^2 B - 4 \cot C \cot A - 2 \cot A \cot B - 2 \cot B \cot C}{2 (\cot A + \cot B + \cot C)}$$

$$= -\frac{1 + \cot^2 B + 2 \cot C \cot A}{2 \cot \omega},$$

since $$\Sigma \cot B \cot C = 1 \quad \text{and} \quad \Sigma \cot A = \cot \omega;$$

therefore $$\cot \omega_2 = 2 (\cot A + \cot B + \cot C) - \frac{1 + \cot^2 B + 2 \cot C \cot A}{2 \cot \omega}$$

$$= 2 \cot \omega - \frac{\operatorname{cosec}^2 B + 2 \cot C \cot A}{2 \cot \omega}.$$

Similarly the rest. Therefore

$$\Sigma \cot \omega_r = 6 \cot \omega - \frac{\Sigma \operatorname{cosec}^2 A + 2 \Sigma \cot B \cot C}{2 \cot \omega} = 6 \cot \omega - \frac{\operatorname{cosec}^2 \omega + 2}{2 \cot \omega},$$

since $$\Sigma \operatorname{cosec}^2 A = \operatorname{cosec}^2 \omega.$$

Therefore $$2 \cot \omega (\Sigma \cot \omega_r) = 12 \cot^2 \omega - \operatorname{cosec}^2 \omega - 2.$$

15771. (A. Holm, M.A.)—Find r rational numbers such that, if a given number is added to their sum or to the sum of any $r-1$ of them, the results may all be squares.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.;
(II.) *by* B. C. Wallis, F.C.P., B.Sc. (Econ.).

(I.) In the present solution the given number (N) is taken to be an integer. In this case the solution can be effected in integers. Let $x_1, x_2, \ldots, x_r$ be the r sought numbers; s their sum. Then $s + N = \sigma^2$,

$$(s - x_1) + N = \sigma_1^2; \quad \text{therefore} \quad x_1 = \sigma^2 - \sigma_1^2;$$
$$(s - x_2) + N = \sigma_2^2; \quad \text{therefore} \quad x_2 = \sigma^2 - \sigma_2^2;$$
$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots$$
$$(s - x_r) + N = \sigma_r^2; \quad \text{therefore} \quad x_r = \sigma^2 - \sigma_r^2;$$

therefore $$s = r\sigma^2 - \Sigma_1^r \sigma_r^2, \quad \text{whence} \quad \Sigma_1^r \sigma_r^2 = (r-1)\sigma^2 + N.$$

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The quantity σ and all but four (say σ_1, σ_2, σ_3, σ_4) of σ_r may now all be assumed at pleasure, subject only to the condition

$$\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2=(r-1)\sigma^2+N-\Sigma_5^r\sigma_r^2=a+\text{quantity}.$$

This is always possible in many ways. From this equation σ_1, σ_2, σ_3, σ_4 can always be determined in integers (in at least one way). Hereby all the r quantities σ_r become known. It remains to be shown that their sums really $= s$. Suppose their sum to be s'. Then

$$s'=r\sigma^2-\Sigma_1^r\sigma_r^2=s \text{ (from above)}.$$

(II.) Let a, b, c, d, ... be such a series; and let the number x added to their sum yield S^2, and let $S^2-a=m^2$ and $S^2-b=n^2$, ...; then, if $S-1=m$ and $S-2=n$, ..., $a=2S-1$, $b=(2S-1)+(2S-3)$, ... using the first differences; *i.e.*, $a=2S-1$, $b=4S-4$, $c=6S-9$, the p-th term $=2pS-p^2$, and the sum of the series

$$=(2S+4S+6S+8S+\ldots)-(1+4+9+16+\ldots)$$

which (to k terms) $=k(k+1)[S-\frac{1}{3}(2k+1)]$;

therefore $$S^2-x=k(k+1)[S-\tfrac{1}{3}(2k+1)];$$

e.g., if $S=13$, then $a=25$, $b=48$, $c=69$, and $x=27$. Similarly, the series 57, 112, 165, 216 will, if 291 be added, equal 29^2. Thus, if one starts with any square (say 157^2), k can be found by inspection from the equation above (in this case $k=11$); k is usually near the square root of the number taken. Here the series is:—313, 624, 933, 1240, 1545, 1848, 2149, 2448, 2745, 3040, 3333; while the number to be added is 4431.

Also, if one desires a definite number of terms in the series (say 78), then S may be found from the equation (6237), and the series found. Here $x=629014$. The series is as follows:—

12473	161993	298800	435365	571688	707769	843608
24944	174440	311225	447768	584069	720128	855945
37413	186885	323648	460169	596448	732485	868280
49880	199328	336069	472568	608825	744840	880613
62345	211769	348488	484965	621200	757193	892944
74808	224208	360905	497360	633573	769544	905273
87269	236645	373320	509753	645944	781893	917600
99728	249080	385733	522144	658313	794240	929925
112185	261513	398144	534533	670680	806585	942248
124640	273944	410553	546920	683045	818928	954569
137093	286373	422960	559305	695408	831269	966888
149544						

The Proposer remarks that Solution (II.) is very interesting, but somewhat indirect. x should be any given number, and, $k=r$ being given, the equation $S^2-x=k(k+1)[S-\frac{1}{3}(2k+1)]$ would not in general give a rational value for S.

9807. (Professor G. B. M. Zerr.)—The perpendiculars from the vertices of a triangle upon the central axis (the line which passes through the circum-centre, the orthocentre, the nine-point centre, and the centroid) are proportional to

$$\cos A\sin(B-C),\quad \cos B\sin(C-A),\quad \cos C\sin(A-B),$$

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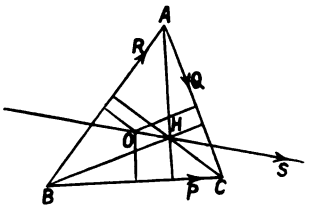
those on one side of the line being reckoned positive, and those on the other negative.

Additional Solution by Professor Sanjána, M.A.

The following statical proof avoids the use of trilinear co-ordinates.

Let forces P, Q, R along the sides BC, CA, AB be of such magnitude as to produce a resultant S along OH, where O is the circum-centre and H the orthocentre. Then the algebraical sum of moments about O vanishes; hence



$$P\cos A + Q\cos B + R\cos C = 0;$$

so also, taking moments about H,

$$P\cos B\cos C + Q\cos C\cos A + R\cos A\cos B = 0.$$

From these two equations we have

$$\frac{P}{\cos A(\cos^2 C - \cos^2 B)} = \frac{Q}{\cos B(\cos^2 A - \cos^2 C)} = \frac{R}{\cos C(\cos^2 B - \cos^2 A)},$$

i.e.,

$$\frac{P}{a\cos A\sin(B-C)} = \frac{Q}{b\cos B\sin(C-A)} = \frac{R}{c\cos C\sin(A-B)}.$$

Again, if P_1, P_2, P_3 be the perpendiculars of the triangle, and π_1, π_2, π_3 the perpendiculars in question, we have, by taking moments about the angular points, $Pp_1 = S\pi_1$, $Qp_2 = S\pi_2$, $Rp_3 = S\pi_3$; so that

$$\begin{aligned}\pi_1 : \pi_2 : \pi_3 &= S\pi_1 : S\pi_2 : S\pi_3 = Pp_1 : Qp_2 : Rp_3\\ &= ap_1\cos A\sin(B-C) : bp_2\cos B\sin(C-A) : cp_3\cos C\sin(A-B)\\ &= \cos A\sin(B-C) : \cos B\sin(C-A) : \cos C\sin(A-B).\end{aligned}$$

15773. (Rev. F. H. Jackson.)—Show that

$$\left(1 - \frac{x}{1^3} + \frac{x^2}{1^3.2^3} - \frac{x^3}{1^3.2^3.3^3} + \dots\right)\left(1 + \frac{x}{1^3} + \frac{x^2}{1^3.2^3.3^3} + \dots\right) = 1 - \frac{3!}{1^3.2^3.1^3}x^2 + \frac{6!}{1^3.2^3.3^3.4^3.1^3.2^3}x^4 - \dots.$$

Solution by the Proposer.

Taking the product of the series,

coefficient of x^n (n odd) $\equiv 0$,

$$\begin{aligned}\text{coefficient of } x^n\ (n \text{ even}) &\equiv \left[\left(\frac{1}{n!}\right)^3 - \left(\frac{1}{(n-1)!\,1!}\right)^3 + \dots + \left(\frac{1}{n!}\right)^3\right]\\ &\equiv \frac{1}{(n!)^3}\left[1 - \left(\frac{n}{1}\right)^3 + \left(\frac{n(n-1)}{2!}\right)^3 - \dots\right]\\ &\equiv \frac{1}{(n!)^3}(-1)^{\frac{1}{2}n}\frac{(\frac{3}{2}n)!}{[(\frac{1}{2}n)!]^3} = (-1)^{\frac{1}{2}n}\frac{(\frac{3}{2}n)!}{[n!\,(\frac{1}{2}n)!]^3}.\end{aligned}$$

The sum of the series of cubes is well known, having been discussed by many writers—Morley, MacMahon, Richmond, Dixon—for general values of n as well as integral values. The series of cubes is due originally to Prof. F. Morley.

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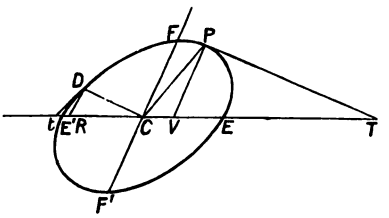
Propositions connected with the Equi-conjugate Diameters of an Ellipse.

By W. F. Beard, M.A.

Prop. 1.—*If* CP, CD *are two conjugate semi-diameters and* PV, DR *are the ordinates to one of the equi-conjugate diameters, then* PV = CR *and* CV = DR.

Let ECE′, FCF′ be the equi-conjugates. Let the tangents at P, D meet ECE′ at T, *t*. Then

VT/CR = PT/CD
(from similar triangles)
= CT/C*t*
(from similar triangles)
= CR/CV,



for CV.CT = CE² = CR.C*t*.

Thus CR² = CV.VT = CV.CT − CV² = CE² − CV² = PV²,

because PV² + CV² = CE², as CE is an equi-conjugate; therefore CR = PV, and similarly DR = CV.

Prop. 2.—*If* PV, PV′ *be the ordinates to the equi-conjugates and* CD *the semi-diameter conjugate to* CP, *and if* PV *produced meet the ellipse at p, then* VV′ = C*p* = CD.

In the triangles VCV′, DRC,

CV = DR
CV′ = PV = CR } from Prop. 1,

and ∠VCV′ = ∠CRD;

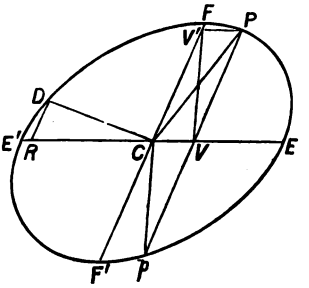
therefore VV′ = CD.

Also *p*V = PV = CV′;

therefore C*p* = VV′,

and is parallel to VV′

= CD.



Prop. 3.—*To prove the sum of the squares on two semi-conjugate diameters is constant.*

In the figure of Prop. 2,

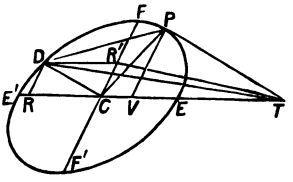
CP² + CD² = CP² + C*p*² = 2(PV² + CV²) = 2CE² = const. = CA² + CB²,

by making P coincide with A.

Prop. 4.—*To prove that the area of the conjugate parallelogram is constant.*

Let PCP′, DCD′ be two conjugate diameters. Let the tangent at P meet ECE′ at T. Draw DR, DR′, the ordinates of D. Join DT. The area of the conjugate parallelogram, *i.e.*, the parallelogram formed by the tangents at P, D, P′, D′,

= 8△PCD = 8△DCT



(because PT is parallel to CD) = 8△R′CT (because DR′ is parallel to CT).

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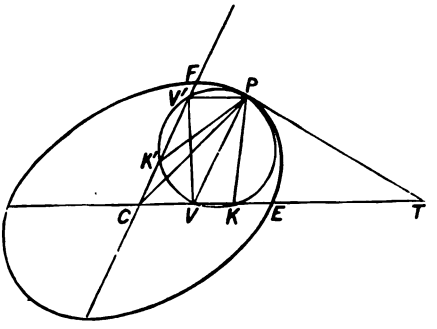
But the area of the triangle R′CT : CR′ . CT is a constant ratio, and

CR′ . CT = DR . CT = CV . CT (because DR = CV) (Prop. 1)

= CE² = constant;

therefore the triangle R′CT, and consequently the conjugate parallelogram, is of constant area.

PROP. 5.—*If* PV, PV′ *are the ordinates to the equi-conjugates, the circle round* PVV′ *touches the ellipse at* P, *and, if it cuts the equi-conjugates at* K, K′, *then* PK = PK′ = CD.



Let the tangent at P to the ellipse meet CE at T. Then

(1) PV² = CE² − CV²

= CV . CT − CV²

= CV . VT

= PV′ . VT;

therefore

PV′ : PV = PV : VT

and ∠PVT = ∠VPV′;

therefore the triangles VPV′, TVP are similar;

therefore

∠VPT = ∠PV′V,

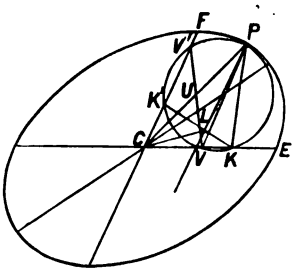
and therefore PT touches the circle at P, *i.e.*, the circle VPV′ touches the ellipse at P.

(2) PV′, VK are parallel chords of the circle; therefore

PK = VV′ = CD (Prop. 2) = similarly PK′.

COR.—If PV produced meets the ellipse at p, then the circle round the triangle CPT passes through p and K′; for pV = PV; therefore PV . Vp = CV . VT; therefore the circle round CPT goes through p. Also ∠PK′V′ = ∠PVV′ (angles in same segment) = ∠PTV, because the triangles PVV′, VTP are similar; therefore the circle round CPT goes through K′.

PROP. 6.—*In the figure of Prop.* 5, *let* KK′ *meet the normal at* P *in* L. *Then* ∠LCE = ∠PCV′ *and* CL : CP *is a constant ratio.*



(1) Let CP meet VV′ at U. PK = PK′ and PL is a diameter of the circle PKK′; therefore PL bisects KK′ at right angles.

∠CKK′ = ∠CV′V

in same segment; therefore the triangles CKK′, CV′V are similar, and CL, CU are medians from C in these two triangles; therefore the triangles LCK, UCV′ are also similar; therefore

∠LCK = ∠UCV′.

(2) From similar triangles,

CL : LK = CU : UV′ = CP : VV′ (because CP, VV′ bisect one another

= CP : CD (from Prop. 2);

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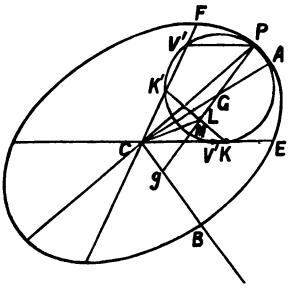
therefore CL : CP = LK : CD = LK : PK (Prop. 5).
But ∠PKK′ = ∠PV′F (because P, K, K′, V′ are concyclic)
= ∠V′CV (by parallels);
therefore the angle PKL is constant and PLK is a right angle; therefore LK : PK = a constant ratio; therefore CL : CP = a constant ratio.

Note that CL : LK = CP : CD.

Cor.—The locus of L is an ellipse similar and similarly situate to the given ellipse.

Prop. 7.—*If the normal at* P *meets the major and minor axes in* G, *g, then*
PG : CD = CD : P*g* = CB : CA.



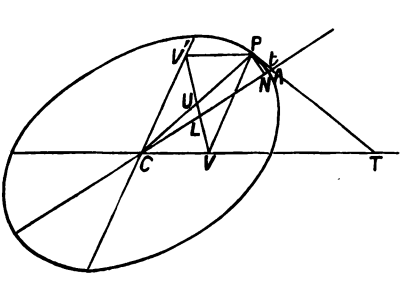
Let CA, CB be the major and minor axes, which bisect the angle ECF internally and externally. ∠LCK = ∠PCV′ (from Prop. 6); therefore CA, CB bisect PCL internally and externally; therefore
PG : GL = CP : CL
= constant ratio (Prop. 6);
therefore
PG : PL = constant ratio,
and PL : CD = constant ratio,
because ∠PKL = ∠ECF (Prop. 6),
and PK = CD (Prop. 5);
therefore PG : CD = constant ratio = CB : CA by making P coincide with B. Similarly P*g* : CD = CA : CB.

Cor.—If the normal at P meets the circle PVV′ at M, M is the mid-point of G*g*.

∠V′KC = ∠PVK (because PV′, VK are parallel chords of the circle)
= ∠ECF (because PV is parallel to CF);
therefore CV′ = KV′, and the angle MV′P is a right angle (in a semicircle); therefore V′M is perpendicular to CK, and bisects CK.

Also CL : CP = LK : KP (Prop. 6) = PG : GL = P*g* : *g*L; therefore C, K both lie on the circle on G*g* as diameter, and therefore V′M, which bisects CK at right angles, passes through the centre of this circle; therefore M is the mid-point of G*g*.

Prop. 8.—*If* PN *is drawn perpendicular to* CA, *then* CN : PV + CV *is a constant ratio, and likewise* PN : PV − CV.



Let VV′ meet CP, CA at U, L, and let the tangent at P meet CA, CE at *t*, T. Draw PZ parallel to CA.

(1) We will prove
*t*P = 2UL.
∠ZLA
= ∠V′CL + ∠CV′L
= ∠TCA + ∠CTP
(Prop. 5, Cor.)
= ∠P*t*C,

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and PZ is parallel to Ct; therefore $Pt = ZL$. But $ZU = UL$, because $CU = UP$ and PZ is parallel to CL; therefore $tP = 2UL$.

(2) Because CL bisects the angle VCV', therefore

$$V'L/VL = CV'/CV = PV/CV;$$

therefore $(PV+CV)/(PV-CV) = (V'L+VL)/(V'L-VL)$
$= (VV'/2UL) = CD/tP$,

because $VV' = CD$ (Prop. 2), and $2UL = tP$;

therefore $(PV+CV)^2/(PV-CV)^2 = CD^2/tP^2 = CA^2/(Ct^2-CA^2)$;

therefore $(PV+CV)^2/[(PV+CV)^2+(PV-CV)^2] = CA^2/Ct^2 = CN^2/CA^2$,

because $CN \cdot CT = CA^2$;

therefore $CN^2/CA^2 = (PV+CV)^2/2(PV^2+CV^2) = (PV+CV)^2/2CE^2$;

therefore $CN^2/(PV+CV)^2 = CA^2/2CE^2 = \text{constant}$;

therefore CN : PV + CV is a constant ratio,
a result which is obvious by projection.

In a similar way we can prove PN : PV − CV a constant ratio, or thus

$$\frac{CN^2}{CA^2} = \frac{(PV+CV)^2}{2CE^2};$$

therefore $$\frac{PN^2}{CB^2} = \frac{CA^2-CN^2}{CA^2} = \frac{2CE^2-(PV+CV)^2}{2CE^2}$$

$$= \frac{PV^2+CV^2-2PV \cdot CV}{2CE^2} \quad (\text{because } 2CE^2 = 2PV^2+2CV^2)$$

$$= \frac{(PV-CV)^2}{2CE^2};$$

therefore PN/(PV − CV) is a constant ratio.

Cor.—If NP meets the auxiliary circle at p,

$$pN/CN = (PV-CV)/(PV+CV),$$

having regard to the sign of CV.

Prop. 9.—*The locus of the intersection of tangents at right angles is a circle.*

Note.—Prop. 9 cannot be omitted in any series of theorems on the non-focal properties of an ellipse, and is therefore inserted, though the proof does not depend on any property of the equi-conjugates.

Let TQ, TQ' be two perpendicular tangents. Join CT, cutting the ellipse at P, and QQ' at V. Let CD be the semi-diameter parallel to QQ'.

$$QV^2 : CP^2-CV^2 = CD^2 : CP^2;$$

therefore $$VT^2 : CV \cdot CT-CV^2 = CD^2 : CV \cdot CT;$$

for QV = VT, since QTQ' is a right angle, and QV = Q'V, and $CP^2 = CV \cdot CT$; therefore

$$VT^2 : CV \cdot VT = CD^2 : CV \cdot CT,$$
$$VT : CV = CD^2 : CV \cdot CT;$$

therefore $$CD^2 = CT \cdot VT, \quad CP^2 = CT \cdot CV;$$

therefore $$CT^2 = CP^2+CD^2 = CA^2+CB^2;$$

therefore the locus of T is a circle.

N.B.—This proof has already appeared in the *Mathematical Gazette*.

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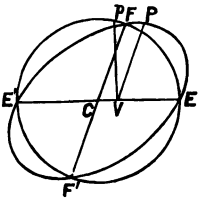
PROP. 10.—*If from the foot of the ordinate to an equi-conjugate a straight line be drawn at right angles to the equi-conjugate and equal to the ordinate, the locus of its extremity is the circle through the ends of the equi-conjugates.*

Let PV be the ordinate of P to CE, and let Vp be drawn perpendicular to CE and equal to PV. Then

$$Cp^2 = CV^2 + pV^2 = CV^2 + PV^2 = CE^2;$$

therefore $Cp = CE$, and the locus of p is the circle through EFE'F'.



Additional Remarks.

Prop. 10 suggests an interesting method of treating the ellipse.

Define an ellipse as the locus of a point such that the sum of the squares of its ordinates to two oblique lines is constant.

Let CE, CF be the two given oblique lines.

Definitions.—If from any point p a perpendicular pV be drawn to CE, and VP is drawn parallel to CF and equal to pV, then p, P are said to be *co-points* with regard to CE.

The circle centre C, such that the square of its radius is equal to the constant quantity in the definition of the ellipse, is called the *equi-cycle.*

Proofs of the following propositions are not given, but are quite simple:—

Prop. 1.—The line joining two co-points is fixed in direction.

Prop. 2.—If from any two points P, Q parallel lines are drawn to cut CE at M, N, and p, q are the co-points of P, Q, then pM, qN are also parallel.

Prop. 3.—If a point moves on a fixed straight line, then its co-point also moves on a fixed straight line.

Prop. 4.—The co-point of any point on the equi-cycle lies on the ellipse.

Prop. 5.—Let o, O be any two co-points, T any point on CE. Let oT cut the equi-cycle in p, q; then will OT cut the ellipse in the co-points of p, q.

Prop. 6.—The locus of the middle points of parallel chords of an ellipse is a straight line through the centre.

Prop. 7.—If the tangent at P meet CE at T, and if PV be the ordinate of P to CE, then $CV \cdot CT = CE^2$.

Prop. 8.—If from a fixed point straight lines be drawn to cut the ellipse, the rectangle contained by the segments of each varies as the square of the parallel semi-diameter.

N.B.—The other central properties of the ellipse will now follow as proved in the previous propositions.

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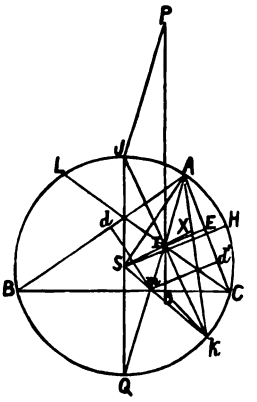
15729. (R. Tucker, M.A.)—The circle $dd'k$ touches AB, AC, and the arc BC of the circle ABC (internally); the circle $ee'k'$ touches BC, BA, and the arc CA; and the circle $ff'k''$ touches CA, CB, and the arc AB. Prove (i.) ed', fe', df' are parallel to AB, BC, CA respectively; (ii.) $Cd' . Ae' . Bf' = Af . Bd . Ce$; (iii.) ρ_1 (radius of circle $dd'k$) $= r \sec^2 \frac{1}{2}A$; (iv.) Ak, Bk', Ck'' intersect in a point.

Another Solution by H. W. Webster, M.D.

Let ABC be any triangle. Bisectors of A, B, and C meet the circum-circle in Q, H, L. The diameter QSJ is perpendicular to BC. The in-radius DI is continued to P, making IP = QJ. Then JQIP is a parallelogram. JI meets the circum-circle in k. The radius Sk cuts AI in m, and md' is perpendicular to AC, and therefore parallel to SH and to IE, the in-radius. md is perpendicular to AB.



(i.) Because

$$2Rr = AI . IQ = JI . Ik = IP . ID,$$

the points J, P, D, k are cyclic; therefore the angles DkI, JPI, JQI, IkA, SAQ are equal. Also mkI, SJk, mAk, DIk are equal; therefore IDk, AIk; Amk, kmI; md'A, IEA are three pairs of similar triangles. By multiplying the ratios

$$\frac{ID}{Ik}\,\frac{Am}{Ak}\,\frac{md'}{mA} = \frac{AI}{Ak}\,\frac{km}{kI}\,\frac{IE}{IA}$$

or $md' = mk$; therefore d, d', k are points touched by the circle, with centre m.

Because $km/md' = k$S/SH, the points k, d', H are collinear: and similarly the points k, d, L; therefore the points B A, C, L, k, H form a Pascal's hexagon, and the intersections d', I, d are collinear; and obviously AI bisects dd' at right angles. Similarly it is shown that I bisects ee' and ff'; therefore ee' and ff' form a parallelogram, and $e'f$ is parallel to BC. So with others.

(ii.) Because $Ae'f$, ABC are similar, $Ae'/AB = Af/AC$. By multiplying these and the corresponding ratios from B and C, the required result is obtained.

(iii.) $md'/IE = mA/IA = \triangle md'A/\triangle d'IA = mA^2/d'A^2 = \sec^2 \frac{1}{2}A$.

(iv.) The triangle SIm and the transversal AkX will give, by theorem of Menelaus, $(SX . IA . mk)/(Sk . IX . mA) = 1$; therefore

$$SX/IX = Sk/mk . mA/IA = Sk/mk . md'/IE = R/r,$$

similarly Bk' and Ck'' can be shown to cut SI in ratio $R : r$. Therefore they meet in a point.

15770. (C. E. Youngman, M.A.)—A sum of money is made up of ordinary coins of the realm, each of as large value and as little frequency

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as may be consistent with the money being shared as follows by A., B., and C.:—first A. takes a penny and one-third of the rest, then B. a penny and one-third of the rest, then C. likewise, and the remainder they share equally. Then one of them discovers that his coins are worth a shilling each on the average. Find (without algebra) who this is.

Solutions (I.) *by* B. C. Wallis, F.C.P., B.Sc. (Econ.); (II.) *by the* Proposer.

(I.) C. has a total of 8s. in 8 coins out of a total of 32s. 5½d. Since each takes a third after taking a penny, what he takes is half of what he leaves. For convenience of nomenclature call the amount each received at the final division x. Then C. receives 1d., $1\frac{1}{2}$ times x, and x; B. receives 1d., $2\frac{1}{4}$ times x plus ¼d., and x; A. receives 1d., $3\frac{3}{8}$ times x plus $1\frac{1}{4}$d., and x. That is, the second amounts taken in each case involve for A. $1\frac{1}{4}d.$, for B. ¼d., and for C. no odd fractions of a penny. Each man must take a penny, and the one who has an average value of a shilling must also have n coins to the value of ns. and 11d., *i.e.*, he must have the original penny, two half-crowns, one florin, a sixpence, a three-penny piece, and two pennies: a total of eight coins to the value of 8s.

Consider the case of A.: if he had 8s., $4\frac{3}{8}$ times x = 8s. less 2¼d., which yields a value for x of $87\frac{5}{7}$ farthings, an impossible value. If B. had 8s., then $3\frac{1}{4}$ times x = 8s. less 1¼d., from which $x = 116\frac{4}{13}$ farthings. In the case of C., $2\frac{1}{2}$ times x = 8s. less 1d., whence x = 38 pence. Therefore the total is 32s. 5½d. A. receives 1d., 10s. 9½d., and 3s. 2d.; in all 14s. 0½d. B. receives 1d., 7s. 2d., and 3s. 2d., in all 10s. 5d. C. receives 1d., 4s. 9d., and 3s. 2d.; in all 8s.

(II.) Add 2d. to the money; then let A. take $\frac{1}{3}$, leaving $\frac{2}{3}$; B. $\frac{2}{5}$, leaving $\frac{4}{5}$; C. $\frac{4}{7}$, leaving $\frac{8}{7}$; and give each in addition $\frac{8}{7}$. Thus A. gets $\frac{3}{7}\frac{8}{1}$, B. $\frac{3}{5}\frac{2}{1}$, and C. $\frac{3}{7}\frac{2}{1}$; and the result is the same as in the Question, except that each has ¾d. too much. One or other, then, of these shares is worth certain exact shillings, + ¾d.; in other words, it is ¼ of 3s. 2d., or 6s. 2d., or 9s. 2d., ... But the smallest possible number of coins (averaging 1s. each) in any share is 6, viz., three pennies, a 3d., a 6d., and a 5s.; and even this will not do—the 5s. is too much all at once, because no one gets more than about $\frac{3}{7}\frac{7}{5}$ of his total at one taking. Hence in the series above we need not consider terms less than 21s. 2d.; so that in farthings the series will be 1016, 1160, 1304, ...; in which we have to find a term divisible by 35 or 26 or 20. Plainly the 1160 (= 24s. 2d.) is divisible by 20; indicating 8s. as C.'s share, and $\frac{8}{3}\frac{1}{0}$ of ⅓ of 24s. 2d., or 32s. 7¼d., as the original sum (with the added 2d.). This, tested by sharing it according to the Question, turns out right; so that it is perhaps unnecessary to seek terms divisible by 35 or 26, or again by 20. They are easily found, but result in shares so large that the coins composing them would average more than a shilling each.

15784. (R. F. Davis, M.A. Suggested by Question 15748.)—If ABCD be a quadrilateral whose opposite sides BA, CD meet in E, and

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any variable transversal EPQ meet AD, BC in P, Q respectively, then $AP.PD/EP^2 : BQ.QC/EQ^2$ is constant.

Solutions (I.) *by* W. Austin Sleigh, B.A., *and* H. W. Webster, M.D.;
(II.) *by* W. J. Dobbs, M.A.; (III.) *by* J. Blaikie, M.A.

(I.) Let AD, BC meet in F. Because the sides of the triangle FPQ are cut by the transversals EDC, EAB respectively, we have (Todhunter's *Euclid*, Appendix 57) $PE.QC.FD = EQ.CF.DP$, *i.e.*,

$$(PE.QC)/(EQ.DP) = CF/FD \quad \ldots\ldots (1),$$

and $PE.QB.FA = EQ.BF.AP$, *i.e.*,

$$(PE.QB)/(EQ.AP) = BF/FA \quad \ldots\ldots (2).$$

Combining (1) with (2), we have

$$(AP.PD/PE^2)/(BQ.QC/EQ^2) = (AF.FD)/(BF.FC).$$

The right-hand side is a constant for all positions of EPQ.

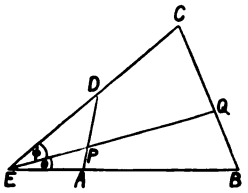
(II.)

$(AP/EP)(PD/EP)$
$= (\sin\theta/\sin A)(\sin\phi/\sin D).$

Also $(BQ/EQ)(QC/EQ)$
$= (\sin\theta/\sin B)(\sin\phi/\sin C)$;

therefore

$AP.PD/EP^2 : BQ.QC/EQ^2$
$= \sin B \sin C : \sin A \sin D.$



(III.) Draw PM, QN perpendicular to EB.

$$AP \sin A/BQ \sin B = PM/QN = EP/EQ;$$

therefore $AP/EP : BQ/EQ = \operatorname{cosec} A : \operatorname{cosec} B.$

Similarly $DP/EP : CQ/EQ = \operatorname{cosec} D : \operatorname{cosec} C;$

therefore $AP.DP/EP^2 : BQ.CQ/EQ^2 = \operatorname{cosec} A \operatorname{cosec} D : \operatorname{cosec} B \operatorname{cosec} C$
$= a$ constant ratio.

15791. (A. H. Bell.)—When is a triangular number a pentagona number? Required, a general solution with examples.

Solution by J. Blaikie, M.A., *and* F. N. Mayers, M.A.

The general expression for a triangular number is $N = \frac{1}{2}n(n+1)$ and for a pentagonal number $N = \frac{1}{2}m(3m-1)$. We have therefore to find positive integral solutions of the equation

$$n^2+n = 3m^2-m \quad \ldots\ldots (1).$$

This may be written in the form

$$3y^2-x^2 = 2 \quad \ldots\ldots (2),$$

where $x = 6m-1, \quad y = 2n+1.$

If p and q be values of x and y which satisfy the equation

$$x^2-3y^2 = 1 \quad \ldots\ldots (3),$$

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then the values $x = 143q \pm 71p$, $y = 41p \pm pq$ satisfy (2). For

$$3y^2 - x^2 = 3(41p \pm 71q)^2 - (3 \cdot 41q \pm 71p)^2$$
$$= (3 \cdot 41^2 - 71^2)(p^2 - 3q^2) = 2 \cdot 1 = 2.$$

The values of p and q which satisfy (3) are the alternate convergents to $\sqrt{3}$, viz., 1, 0; 2, 1; 7, 4; 26, 15; 97, 56; ...; but these only give integral values of m and n when $x+1$ is divisible by 6 and $y-1$ by 2. These conditions are satisfied in the 1st, 3rd, and 5th cases taking the positive sign, when we get the values given in the adjoining table.

p	q	x	y	m	n	N
1	0	71	41	12	20	210
7	4	989	571	165	285	40755
97	56	13775	7953	2296	3976	7906276

11089. (Professor Sylvester.)—Let $x_1, x_2, \ldots, x_n$ represent $x+a_1$, $x+a_2, \ldots, x+a_n$, where $a_1, a_2, \ldots, a_n$ are any *real* quantities. Also let $b_1, b_2, \ldots, b_n$ be any positive quantities. Prove that the continued fraction

$$x_1 - \frac{b_1}{x_2 -} \frac{b_2}{x_3 -} \cdots \frac{b_{n-1}}{-x_n}$$

changes sign exactly $2n-1$ times as x passes from one end of infinity to the other.

Solution by Professor Nanson.

Let y_n denote the continued fraction, and let

$$z_{n-1} = x_2 - \frac{b_2}{x_3 -} \frac{b_3}{x_4 -} \cdots \frac{b_{n-1}}{-x_n},$$

so that $y_n = x_1 - b_1/z_{n-1}$. Now assume that as x increases from $-\infty$ to $+\infty$ dy_n/dx is positive, and y_n passes n times from $-\infty$ to $+\infty$ and therefore vanishes n times, viz., once in each interval, and consequently changes sign $2n-1$ times, viz., once in each of the n intervals and once at each of the $n-1$ boundaries between two intervals. The like properties therefore hold for z_n. But we have

$$y_{n+1} = x_1 - b_1/z_n, \quad dy_{n+1}/dx = 1 + (b_1/z_n^2)\, dz_n/dx.$$

Hence dy_{n+1}/dx is positive; so that y_{n+1} increases with x. Also y_{n+1} changes from $+\infty$ to $-\infty$ as x increases through each of the n values which make z_n vanish. Further, y_{n+1} is $-\infty$ when $x = -\infty$ and is $+\infty$ when $x = +\infty$. Hence as x passes from $-\infty$ to $+\infty$ y_{n+1} passes $n+1$ times from $-\infty$ to $+\infty$ and increases throughout each interval. It follows that y_{n+1} vanishes $n+1$ times, viz., once in each interval, and consequently changes sign $2n+1$ times, viz., once in each of the $n+1$ intervals and once at each of the n boundaries between two intervals. Thus, if the theorem be true for y_n, it is true for y_{n+1}. But inspection of the graph of y_2 shows that the theorem is true for y_2. Therefore, &c.

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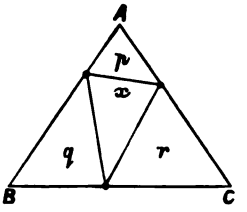
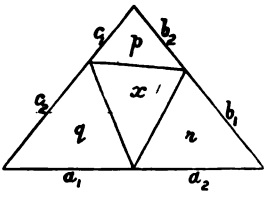
15769. (R. CHARTRES.)—A random point being taken in each side of the triangle ABC, find the mean value of $p^h q^k r^l$, x^n, and $\log x$. Elementary proof required.



Solution by R. CHARTRES.

Let $\Delta = 1$.

$$\begin{aligned} &\mathrm{M}(p^h q^k r^l) \\ &= \mathrm{M}\left[(c_1 b_2)^h (c_2 a_1)^k (a_2 b_1)^l\right]/(bc)^h (ca)^k (ab)^l] \\ &= \mathrm{M}\,(a_1^k a_2^l b_1^l b_2^h c_1^h c_2^k)/a^{k+l} b^{l+h} c^{h+k} \\ &= \frac{(h!\,k!\,l!)^2}{(h+k+1)!\,(k+l+1)!\,(l+h+1)!} \qquad \ldots\ldots(\mathrm{A}), \end{aligned}$$



since $a^{k+l} = (a_1 + a_2)^{k+l}$, having $(k+l+1)$ terms of equal mean value.

If, $y = p + q + r$, then, from (A),

$$\mathrm{M}(y) = \tfrac{2}{4}, \quad \mathrm{M}(y^2) = \mathrm{M}(3p^2 + 6pq) = \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12},$$
$$\mathrm{M}(y^3) = \mathrm{M}(3p^2 + 18p^2 q + 6pqr) = \tfrac{6}{12}\tfrac{7}{4}, \ \ldots;$$

therefore $\mathrm{M}(x^n) = \mathrm{M}(1-y)^n = 1 - \tfrac{n}{4}\mathrm{C}_1 + \tfrac{7}{12}\mathrm{C}_2 - \tfrac{6}{12}\tfrac{7}{4}\mathrm{C}_3 + \tfrac{3}{6}\tfrac{3}{0}\tfrac{7}{4}\mathrm{C}_4 - \ldots,$

$$\mathrm{M}(\log x) = -\left(\tfrac{3}{4} + \tfrac{1}{2} \cdot \tfrac{7}{12} + \tfrac{1}{3} \cdot \tfrac{6}{12}\tfrac{7}{4} + \ldots\right),$$

putting C_r for C_r^n.

If we express (A) in gamma functions, the result will hold for all values: thus, if $h = -\frac{1}{2} = k = l$, $\mathrm{M}(p^{-\frac{1}{2}} q^{-\frac{1}{2}} r^{-\frac{1}{2}}) = \pi^2$.

15776. (Professor HUDSON, M.A.) — Prove that the cusps of the evolute of the focus-roulette of a hyperbola on a straight line are at the same distance from that line.

Solution by the PROPOSER.

The cusps of the evolute are the centres of maximum and minimum curvature of the roulette; from symmetry, these occur when the vertices are in contact with the straight line. If $2a$ be the transverse axis, e the eccentricity, ρ the radius of curvature at the vertex of the hyperbola, R, R′ the radii of curvature of the roulette when the nearer and further vertices are in contact, we have (Besant's *Roulettes and Glissettes*, Art. 30) $\mathrm{R} = r^2/(r-\rho)$, $\mathrm{R}' = r'^2/(r'-\rho')$, where $r = a(e-1)$, $r' = a(e+1)$, $\rho = -\rho' = a(e^2-1)$. The distances of the centres of curvature from the straight line are $r - \mathrm{R}$, $r' - \mathrm{R}'$, and each of these is $a(e - 1/e)$. If r, R be any focal distance and the corresponding radius of curvature of the roulette, the relation $1/\mathrm{R} + 1/r = \text{constant}$ (Art. 31) shows that the evolute has no other cusps.

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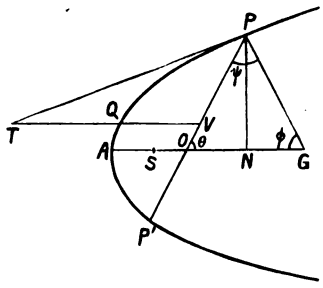
15726. (Professor SANJANA, M.A.)—Show that the normals drawn at the extremities of any chord of a parabola and terminated by the axis have equal projections upon that chord; and that these projections are constant when the chord moves so that the algebraical difference of the ordinates of the two extremities has a constant projection on the chord. Prove also that this condition is satisfied by every focal chord.

Another Solution by M. V. A. SASTRY, B.A.

Let POP′ be a chord of the parabola making an angle θ with the axis and an angle ψ with the normal PG. Let PG make an angle ϕ with the axis. Let the tangent at P meet the diameter QV at T.



Now, from the triangle PTV,

$$\cos\psi : \cos\phi = 2QV : PV;$$

therefore

$$PV\cos\psi = 2QV\cos\phi.$$

But $4QS \cdot QV = PV^2$;

therefore $2QS\cos\psi = PV\cos\phi$. But $QS = a\,\mathrm{cosec}^2\theta$, as can be easily shown; therefore $\cos\psi = (PP'\cos\phi \cdot \sin^2\theta)/4a$; therefore the projection of PG on PP′

$$= PG\cos\psi = PP'\sin^2\theta \cdot PG\cos\phi/4a$$
$$= PP'\sin^2\theta \cdot NG/4a \quad \text{(where PN is the ordinate of P)}$$
$$= \tfrac{1}{2}PP'\sin^2\theta \quad \text{(because } NG = 2a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

The projection of the normal at P′ on PP′ can be shown by an exactly similar method to be equal to the same expression $\frac{1}{2}PP'\sin^2\theta$. Hence the first part of the theorem follows.

Now, the projection of PN on PP′

$$= PN\sin\theta = PO\sin^2\theta,$$

and similarly the projection of P′N′ on PP′

$$= P'O\sin^2\theta;$$

therefore the algebraic difference of the projections of the ordinates on the chord $= PP'\sin^2\theta$ ..... (B);

therefore, if this difference is constant, it follows from (A) that the projections of PG and P′G on PP′ are each constant. Hence the second part of the theorem.

In the case of a focal chord, it is obvious that the projections of the normals on the chord are each equal to $2a$. Hence in this case the condition is satisfied.

15713. (Professor LAUVERNAY.)—Dans un triangle rectangle ABC, α, β, γ sont les points de contact de cercles ex-inscrits respectivement avec les côtés BC, CA, AB. Démontrer que, si les droites $A\alpha$, $B\beta$, $C\gamma$

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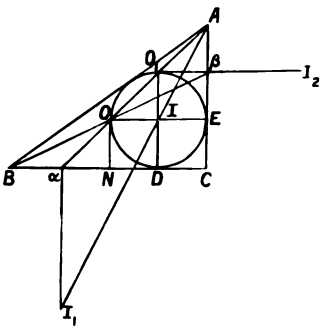
sont concourantes sur la circonférence inscrite à ce triangle, les côtés de ce triangle sont proportionnels aux nombres 3, 4, 5.

Geometrical Solution by W. F. Beard, M.A.



Let I be the in-centre; I_1, I_2, I_3 the ex-centres. Let the in-circle touch the sides at D, E, F. Let $A\alpha$, $B\beta$ meet at O, and let AO meet the in-circle again at O_1. Draw ON perpendicular to BC. A is the centre of similitude of the in-circle and the ex-circle opposite A. $A\alpha$ meets the two circles at O_1, α; therefore IO_1 is parallel to $I_1\alpha$, and therefore perpendicular to BC; therefore O_1I passes through D.

Similarly OI is parallel to $I_2\beta$, and therefore passes through E. Let the radius of the in-circle $= r$; then $DC = IE = r$, $ND = OI = r$. Also $ON/\alpha N = O_1I/OI$ from similar triangles, but $O_1I = OI$; therefore $\alpha N = ON = r$, and $B\alpha = CD = r$; therefore $a = 4r$. $\alpha N = ND = DC$; therefore $\alpha O = OO_1 = O_1A$; therefore $CE = E\beta = A\beta$; whence $b = 3r$. $c^2 = a^2 + b^2$, given; therefore $c^2 = 25r^2$, $c = 5r$. Thus $a : b : c = 4 : 3 : 5$.

N.B.—We ought to prove that OI cannot be perpendicular to BC.

15800. (R. F. Davis, M.A.)—Prove that the semi-axes of the ellipse which is inscribed in the triangle of reference and has for its centre the point whose trilinear co-ordinates are α, β, γ are determined by the equation $$a\sqrt{(\alpha^2-\rho^2)} + b\sqrt{(\beta^2-\rho^2)} + c\sqrt{(\gamma^2-\rho^2)} = 0.$$

Solution by C. M. Ross.

Let α', β', γ' be the co-ordinates of the centre. Then, if ρ be a semi-diameter drawn from the centre to (α, β, γ),

$$\rho^2 = -(abc/4\Delta^2)\,[a(\beta'-\beta)(\gamma'-\gamma) + b(\gamma'-\gamma)(\alpha'-\alpha) + c(\alpha'-\alpha)(\beta'-\beta)] \quad (1).$$

Again, the equations $\alpha'/(nb+mc) = \beta'/(lc+na) = \gamma'(ma+lb)$ give the centre of the ellipse $\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0$. Now the semi-axes are the greatest and least values of the semi-diameter. Hence (1) must be a maximum or a minimum, α, β, γ being connected by the equations

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0, \quad a\alpha + b\beta + c\gamma = 2\Delta = a\alpha' + b\beta' + c\gamma' \quad (2, 3).$$

Differentiating,

$$ad\alpha + bd\beta + cd\gamma = 0 \text{ and } \sqrt{(l/\alpha)}\,d\alpha + \sqrt{(m/\beta)}\,d\beta + \sqrt{(n/\gamma)}\,d\gamma = 0.$$

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Hence $a\surd(\alpha/l) = b\surd(\beta/m) = c\surd(\gamma/n) = \lambda$; therefore $a\surd\alpha = \lambda\surd l$, $b\surd\beta = \lambda\surd m$, $c\surd\gamma = \lambda\surd n$. Squaring each equation, and adding, $\lambda^2(l/a + m/b + n/c) = a\alpha + b\beta + c\gamma = 2\Delta$ by equations (2), (3). Now, from equations connecting co-ordinates of centre, $2lbc = \mu(b\beta' + c\gamma' - a\alpha')$, $2mca = \mu(c\gamma' + a\alpha' - b\beta')$, $2nab = \mu(a\alpha' + b\beta' - c\gamma')$. By addition,

$$\mu = (lbc + mc + nab)/\Delta$$

by (3). Hence $\beta' - \beta = \beta' - (c\gamma' + a\alpha' - b\beta')/b = (2b\beta' - c\gamma' - a\alpha')/b$,

$$\gamma' - \gamma = (2c\gamma' - a\alpha' - b\beta')/c, \quad \alpha' - \alpha = (2a\alpha' - b\beta' - c\gamma')/a,$$

after reduction; therefore

$$\rho^2 = -\Sigma[a^2(2b\beta' - c\gamma' - a\alpha')(2c\gamma' - a\alpha' - b\beta')]/4\Delta^2.$$

Squaring this equation and transforming, the result is

$$a\surd(\alpha'^2 - \rho^2) + b\surd(\beta'^2 - \rho^2) + c\surd(\gamma'^2 - \rho^2) = 0,$$

the required equation.

The Proposer's solution is as follows:—Let d, e, f be the trilinear co-ordinates of one focus; then ρ^2/d, ρ^2/e, ρ^2/f are the trilinear co-ordinates of the other focus. Since the two foci are equidistant from the centre, $d + \rho^2/d = 2a$, &c. Thus $d = a \pm \surd(a^2 - \rho^2)$; and

$$a\alpha + b\beta + c\gamma = 2\Delta = ad + be + cf,$$

whence the required result.

Note on Pfaffians with Binomial Elements.

By Thomas Muir, LL.D.

1. A proof was recently given by Prof. Nanson (*Reprint*, New Series, Vol. VII., p. 97) of the theorem that

$$\begin{vmatrix} a_1+a_2 & a_1+a_3 & a_1+a_4 & \dots & a_1+a_{2n} \\ & a_2+a_3 & a_2+a_4 & \dots & a_2+a_{2n} \\ & & a_3+a_4 & \dots & a_3+a_{2n} \\ & & \dots & \dots & \dots \\ & & & & a_{2n-1}+a_{2n} \end{vmatrix} = 2^{n-1}(a_1a_3a_5 \dots a_{2n-1} + a_2a_4a_6 \dots a_{2n}).$$

It seems worth noting in addition that by putting $a_1 = 0$ in this we deduce

$$\begin{vmatrix} a_2 & a_3 & a_4 & \dots & a_{2n} \\ a_2+a_3 & a_2+a_4 & \dots & a_2+a_{2n} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 2^{n-1}(a_2a_4a_6 \dots a_{2n}) \quad \dots\dots(\text{A}),$$

and that from the deduced result and the original there is obtained by subtraction

$$\begin{vmatrix} a_1 & a_1 & a_1 & \dots & a_1 \\ a_2+a_3 & a_2+a_4 & \dots & a_2+a_{2n} \\ \dots & \dots & \dots & \dots \end{vmatrix} = 2^{n-1}(a_1a_3a_5 \dots a_{2n-1}) \quad \dots(\text{B}).$$

The suggestion thus arises to establish independently the two derived identities (A) and (B), and then by addition to arrive at the original.

2. If the Pfaffian in (A) be denoted by $(a_2, a_3, a_4, \dots, a_{2n})$, we can show by the elementary laws of Pfaffians that

$$(a_2, a_3, \dots, a_{2n}) = 2a_2(a_4, a_5, \dots, a_{2n}),$$

whence (A) immediately follows. For, subtracting the first frame line

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from the second, and thereafter a_3 times the second from the third, and then the third from the first, we have

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2+a_3 & a_2+a_4 & a_2+a_5 & a_2+a_6 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_2 & a_2 & a_2 & \\ & & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= a_2 \begin{vmatrix} 1 & a_3 & a_4 & a_5 & a_6 \\ 1 & 1 & 1 & 1 & \\ & a_3+a_4 & a_3+a_5 & a_3+a_6 & \\ & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = a_2 \begin{vmatrix} 1 & . & a_4 & a_5 & a_6 \\ 1 & 1 & 1 & 1 & \\ & & a_4 & a_5 & a_6 \\ & & & \cdots & \cdots \end{vmatrix}$$

$$= a_2 \begin{vmatrix} 2 & . & . & . & \\ 1 & 1 & 1 & 1 & \\ & a_4 & a_5 & a_6 & \\ & & \cdots & \cdots & \cdots \end{vmatrix} = 2a_2 \begin{vmatrix} a_4 & a_5 & a_6 \\ a_4+a_5 & a_4+a_6 & \\ & a_5+a_6 & \end{vmatrix}.$$

3. A similar theorem exists in reference to the Pfaffian in (B) after the factor a_1 has been withdrawn. For

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ & a_3 & a_4 & a_5 & a_6 \\ & & a_3+a_4 & a_3+a_5 & a_3+a_6 \\ & & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} . & 1 & 1 & 1 & 1 \\ & a_3 & -a_3 & -a_3 & -a_3 \\ & & a_3+a_4 & a_3+a_5 & a_3+a_6 \\ & & & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \begin{vmatrix} . & 1 & 1 & 1 & 1 \\ & 2a_3 & . & . & . \\ & & a_3+a_4 & a_3+a_5 & a_3+a_6 \\ & & & \cdots & \cdots & \cdots \end{vmatrix} = 2a_3 \begin{vmatrix} 1 & 1 & 1 \\ & a_4+a_5 & a_4+a_6 \\ & & a_5+a_6 \end{vmatrix}$$

$$= 2a_3 \begin{vmatrix} 1 & 1 & 1 \\ & a_5 & a_6 \\ & & a_5+a_6 \end{vmatrix}.$$

4. Perhaps all the facts can best be included in the following enunciation:—*If in the Pfaffian whose every place* (r, s) *is filled by* $a_r + a_s$ *the part common to all the elements of any frame line be omitted, the resulting Pfaffian is equal to* $2^{n-1}(a_1 a_3 a_5 \ldots a_{2n-1})$ *or* $2^{n-1}(a_2 a_4 a_6 \ldots a_{2n})$ *according as the number of the frame line is even or odd: and, if the common part be retained and the other parts omitted, the resulting Pfaffian is equal to* $2^{n-1}(a_2 a_4 a_6 \ldots a_{2n})$ *or* $2^{n-1}(a_1 a_3 a_5 \ldots a_{2n-1})$.

5. *The Pfaffian whose every place* (r, s) *is filled by* $a_r - a_s$ *vanishes.*

Here, if, as in the preceding, we partition the Pfaffian into two, it will be found that the whole vanishes because both parts vanish. For, in the first case, by diminishing the last frame line by the one immediately preceding it, we have

$$\begin{vmatrix} a_1 & a_1 & a_1 & a_1 & a_1 \\ & a_2-a_3 & a_2-a_4 & a_2-a_5 & a_2-a_6 \\ & & a_3-a_4 & a_3-a_5 & a_3-a_6 \\ & & & a_4-a_5 & a_4-a_6 \\ & & & & a_5-a_6 \end{vmatrix}$$

$$= a_1(a_5-a_6) \begin{vmatrix} 1 & 1 & 1 & 1 & . \\ & a_2-a_3 & a_2-a_4 & a_2-a_5 & 1 \\ & & a_3-a_4 & a_3-a_5 & 1 \\ & & & a_4-a_5 & 1 \\ & & & & 1 \end{vmatrix},$$

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which latter is evidently equal to zero; and, in the second case, by subtracting the second frame line from the first, we have

$$\begin{vmatrix} -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ a_2-a_3 & a_2-a_4 & a_2-a_5 & a_2-a_6 & \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} -a_2 & -a_2 & -a_2 & -a_2 & -a_2 \\ a_2-a_3 & a_2-a_4 & a_2-a_5 & a_2-a_6 & \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

that is to say, we reduce the second case to the first.

6. Turning to cognate monomial elements, we may note that, *if the general element be $a_r a_s$, the value of the Pfaffian is $a_1 a_2 \dots a_{2n}$*; and, *if the element be a_r/a_s, the value is $(a_1 a_3 \dots a_{2n-1})/(a_2 a_4 \dots a_{2n})$*. These may be made dependent on the proposition that *the Pfaffian whose elements are all positive unity has positive unity for its value*; but every one of the three may be proved independently by showing that, if the last element be deleted, the Pfaffian vanishes.

15777. (Professor Cochez.)—On donne $y^2 = x^3$. On demande le lieu des sommets des angles droits circonscrits à cet cubique.

Solutions (I.) *by* A. M. Nesbitt, M.A., *and* Professor Sanjána, M.A., (II.) *by* J. A. H. Johnston, M.A., *and* C. M. Ross.

(I.) The equation to a tangent may be written $y = mx - \frac{4}{27}m^3$; we thus have to eliminate m from this, and $y = -x/m + 4/27m^3$. We get

$$x = \tfrac{4}{27}(m^2 - 1 + 1/m^2), \quad y = -\tfrac{4}{27}(m - 1/m),$$

each on dividing by $m + 1/m$; whence at once $x = \frac{27}{4}y^2 + \frac{4}{27}$, the locus required.

(II.) The line $y = mx - \frac{4}{27}m^3$ (i.) touches $y^2 = x^3$ for all values of m, and, if (x, y) be assigned, (i.) gives the values of m for the three tangents that can be drawn from (x, y) to the curve. If two of these are at right angles, $1 + m_1 m_2 = 0$, where

$$m_1 + m_2 + m_3 = 0, \quad m_1 m_2 + m_1 m_3 + m_2 m_3 = -\tfrac{27}{4}x, \quad m_1 m_2 m_3 = -\tfrac{27}{4}y,$$

and therefore $m_3(m_1 + m_2) = 1 - \frac{27}{4}x$, $(m_1 + m_2) = -m_3 = -\frac{27}{4}y$. Eliminate m_3 and $(m_1 + m_2)$, and therefore $y^2 = \frac{4}{27}(x - \frac{4}{27})$ is the locus.

15768. (Professor Sanjána, M.A.) — If P, Q be any two points isogonally conjugate with regard to the triangle ABC, then

$$(AP \,.\, AQ)/(AB \,.\, AC) + (BP \,.\, BQ)/(BC \,.\, BA) + (CP \,.\, CQ)/(CA \,.\, CB) = 1.$$

Another Solution by Professor Nanson.

We have identically

$$\Sigma[(P-A)(Q-A)]/[(B-A)(C-A)] = 1,$$

and taking A, B, ... to be the steps from an arbitrary origin to the points A, B, ..., it follows that

$$\Sigma[(AP)(AQ)]/[(AB)(AC)] = 1,$$

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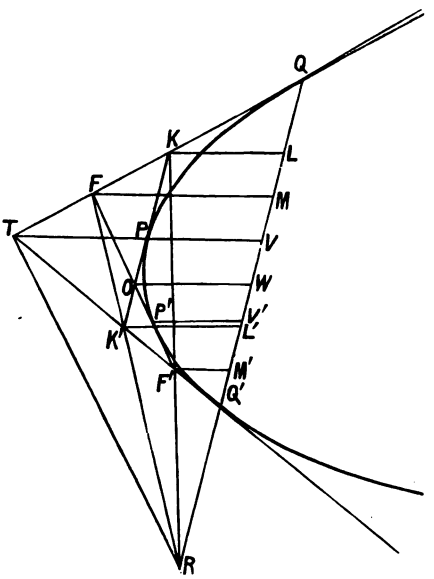
where by (AP) we denote the step from A to P. But, since the angles PAB, QAC are numerically equal and of opposite signs, it follows that we may in this equation replace each step as (AP) by its length and so obtain the required result. *Cf. Reprint*, New Series, Vol. VII., p. 110.

15599 & 15749. (15599.) (A. M. NESBITT, M.A.)—Two tangents drawn from a point T to a parabola meet any third tangent in F and F'. If the diameter through T meet the curve in P, prove that the tangent at P bisects FF'. [Solution by elementary geometrical conics required.]

(15749.) (W. F. BEARD, M.A. Suggested by Question 15599.)—With the figure of Question 15599, if the tangent at P meets TQ, TQ' at K, K' and FK', F'K meet at R, prove that TR is parallel to FF'.

Solution by F. W. REEVES, B.A.



The figure explains itself. I shall first prove Question 15599. Because OP, OP' are tangents and PV, P'V' diameters, therefore

W is mid-point of VV' .....(i.).

Similarly M is the mid-point of QV' and M' of Q'V'; therefore

$VM = QV - QM$
$= \frac{1}{2}(QQ' - QV')$
$= \frac{1}{2}Q'V' = V'M'.$

But $VW = WV'$ from (i.); therefore

$MW = WM'$;

therefore

$FO = OF'$

or FF' is bisected by KK'.

Again, because the quadrilateral KFK'F' circumscribes a circle, therefore the diagonals form a self-conjugate triangle; therefore the polar of T passes through R, *i.e.*, QQ'R is a straight line and KK', which bisects TQ, is parallel to QQ', and therefore bisects TR; therefore TR is parallel to FF', which has been shown to be also bisected by KK'.

Mr. BEARD solves Question 15749 as follows:—In the quadrilateral FK'F'K the mid-points of the diagonals FF', KK' are on KK'; therefore KK' also bisects TR, which is therefore parallel to FF'.

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15523. (A. M. Nesbitt, M.A.)—A man, who has m hats of his own in his hall, is visited by n friends, each wearing a hat. They leave their hats with those of their host. When they are going away they are unfortunately not in a condition to distinguish between one hat and another. Find the chance that no guest takes away his own hat.

Solutions (I.) *by* Professor M. W. Crofton, F.R.S.; (II.) *by the* Hon. G. R. Dick.

There are n objects $a, b, c, \ldots$, each of which has its proper place—say A, B, C, ...—and there are besides m places X, Y, Z, ... If the n objects are distributed at random among the $m+n$ places, to find the chance that no object shall be in its proper place.

It is easy to see that the number of ways the n objects can be distributed is $(m+n)!/m! = (m+1)(m+2)\ldots(m+n)$.

Now this number consists of the cases (1) when *all* the objects are in their proper places, (2) when *all but one* are in their places, (3) when *all but two* are, and so on.

Let u_r = number of ways that r objects, each having a proper place, can be distributed among $m+r$ places (as above) in such a manner that *no object is in its place*.

The number of cases (1) is 1. The number of cases (2) is nu_1: for suppose a the object out of its place: this gives u_1 cases (it is evident that $u_1 = m$); and there are n objects. The number of cases (3) is $\frac{1}{2}[n(n-1)]u_2$: for suppose a, b out of their places: this gives u_2 cases; and the number of such pairs is $\frac{1}{2}[n(n-1)]$; and so on. We infer that

$$(m+1)(m+2)\ldots(m+n) = 1 + nu_1 + \tfrac{1}{2}[n(n-1)]u_2 + \ldots + nu_{n-1} + u_n \ldots (1).$$

By putting n successively equal to 1, 2, 3, ..., we can easily find $u_1, u_2, u_3, \ldots$ from this formula: thus $u_1 = m$, $u_2 = 1 + m + m^2$, ...

To find the general value of u_n, we have, from (1),

$$(m+1)(m+2)\ldots(m+n) + u_0 - 1 = u_0 + nu_1 + \tfrac{1}{2}[n(n-1)]u_2 + \ldots + u_n$$
$$= (1+\mathrm{E})^n u_0,$$

where E is an operator increasing the variable by 1 in any function.

Also $$(m+1)\ldots(m+n-1) + u_0 - 1 = (1+\mathrm{E})^{n-1} u_0,$$
$$(m+1)\ldots(m+n-2) + u_0 - 1 = (1+\mathrm{E})^{n-2} u_0.$$

Now $$u_n = \mathrm{E}^n u_0 = (1+\mathrm{E}-1)^n u_0;$$

hence, from the above values for the powers of $1+\mathrm{E}$,

$$u_n = (m+1)\ldots(m+n) - n(m+1)\ldots(m+n-1)$$
$$+ \tfrac{1}{2}[n(n-1)](m+1)\ldots(m+n-2) - \ldots;$$

therefore $$u_n = \frac{(m+n)!}{m!}\left(1 - \frac{n}{m+n} + \frac{1}{2!}\,\frac{n(n-1)}{(m+n)(m+n-1)} - \ldots\right).$$

Hence, dividing u_n, the number of favourable cases, by the whole number $(m+n)!/m!$, we have for the required probability

$$p = 1 - \frac{n}{m+n} + \frac{1}{2!}\,\frac{n(n-1)}{(m+n)(m+n-1)} - \frac{1}{3!}\,\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)} + \ldots.$$

There are $n+1$ terms. The terms $u_0 - 1$ obviously disappear, as the sum of the binomial coefficients, alternately + and −, is 0.

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Note.—The Problem is a very puzzling one. For the success of the modes of attack used by solvers in earlier volumes of *The Educational Times* and the *Reprint*, it is necessary to form the product of the following probabilities:—(1) the probability of A. not taking his own hat, (2) the probability of B. not taking his own *if* A. *has not*, (3) of C. not doing so *if* A. *and* B. *have not*, and so on; but it may be observed that the difficulty of solving on such lines would probably be very great.

(II.) The probability required may be found by determining the ratio of the number of restricted arrangements to the whole number of possible arrangements. Let $p_{m,n}$ denote the probability, $W_{m,n}$ the number of arrangements satisfying the conditions; then

$$p_{m,n} = W_{m,n}/[(m+n)\dots(m+1)].$$

The problem is the same as:—There are m things of one kind and n of another kind with n corresponding places. In how many ways can the $m+n$ things be arranged in the n vacant places subject to the condition that none of the n things are in their right places?

Let $\theta(x, y)$ denote the number of ways in which y places can be filled up from $x+y$ things $x\,(\leqslant y)$ of one kind all of which are to be used, and y of another kind each of which has a corresponding place and may be put anywhere except in that place; then, evidently,

$$W_{m,n} = \theta(0, n) + m\theta(1, n) + [m(m-1)/(1, 2)]\,\theta(2, n) + \dots + [(m.\,m-1\dots m-\mu+1)/\mu!]\,\theta(\mu, n)\dots,$$

the series terminating when $\mu = m$, if $m \leqslant n$, and when $\mu = n$, if $m \geqslant n$.

(In the original problem none of the host's hats are used, or only one, or only two, &c., or the whole or n as the case may be.)

$$\text{Now } \theta(x, y) = y(y-1)\dots(y-x+1)\{\theta(0, y-x) + x\theta(1, y-x) + [x(x-1)/(1, 2)]\,\theta(2, y-x) + \dots\},$$

wherein $x \leqslant y$, $\theta(z, z) = z!$. Put $\theta(x, y)/(y!) = \phi(x, y)$ and the last relation becomes

$$\phi(x, y) = \phi(0, y-x) + x\phi(1, y-x) + [x(x-1)/(1, 2)]\,\phi(2, y-x) + \dots.$$

We have $p_{m,n} = (m!\,W_{m,n})/(m+n)! = (m!\,n!)\,\phi(m, m+n)/(m+n)!$,

and we have now to determine the form of ϕ applicable to the case. For this purpose, let us put for x successively 1, 2, ... in the equation

$$\phi(x, y) = \phi(0, y-x) + x\phi(1, y-x) + \dots;$$

so $\phi(1, y) = \phi(0, y-1) + \phi(1, y-1)$ or $\phi(1, y) - \phi(1, y-1) = \phi(0, y-1)$.

$\phi(0, n)$ is the probability in the envelope problem and

$$= 1/2! - 1/3! + \dots + (-1)^n/n!.$$

Continuing, we get $\phi(1, y-1) - \phi(1, y-2) = \phi(0, y-2)$, ...,

$$\phi(1, 3) - \phi(1, 2) = \phi(0, 2), \quad \phi(1, 2) - \phi(1, 1) = 0.$$

Thus, by addition,

$$[\phi(r, r) = 1], \quad \phi(1, y) = 1 + \phi(0, 2) + \phi(0, 3) + \dots + \phi(0, y-1).$$

Next, putting $x = 2$, we get

$$\phi(2, y) - \phi(2, y-1) = 1 + \phi(0, 2) + \dots + \phi(0, y-2),$$

and, as before,

$$\phi(2, y) = y - 1 + (y-3)\,\phi(0, 2) + \dots + \phi(0, y-3).$$

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It is easy to continue this process, and we find, by induction,

$$\phi(x,y) = \frac{(y-1)(y-2)(y-x+1)}{1.2.(x-1)} + \frac{(y-3)(y-4)\ \ldots\ (y-x-1)}{1.2.(x-1)}\phi(0,2) + \ldots + \phi(0, y-x);$$

or, writing m for x and $m+n$ for y,

$$\phi(m, m+n) = \frac{1}{(n-1)!}\left[\frac{(m+n-1)!}{n!} + \frac{(m+n-3)!}{(n-2)!}\phi(0,2) + \ldots + \phi(0,n)\right]$$

and $$\frac{m!\,n!}{(m+n)!}\phi(m, m+n)$$

$$= m\left[\frac{1}{m+n} + \frac{n(n-1)}{(m+n)(m+n-1)(m+n-2)}\phi(0,2) + \ldots\right].$$

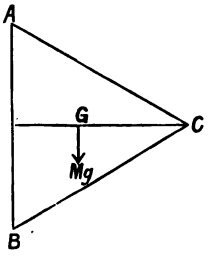
Finally, replacing $\phi(0,n)$ in each case by its value $1/2! - 1/3! + \ldots + (-1)^n/n!$, we get

$$p_{m,n} = \frac{m}{m+n} + \frac{n(n-1)}{2!\,(m+n)(m+n-1)} - \frac{n(n-1)(n-2)}{3!\,(m+n)(m+n-1)(m+n-2)} + \ldots.$$

15786. (I. Arnold.)—An equilateral triangle ABC is suspended at A and its side is kept vertical by a ring at B. It is required to find what the angular velocity of the triangle round AB must be in order that there may be no pressure at B.

Solution by W. J. Dobbs, M.A.

The resultant weight Mg, through the centre of mass G, and the reaction at A must be together equivalent to $M\omega^2(a/2\sqrt{3})$ along CG, where a is the side of the triangle and M its mass. Hence, taking moments about the axis through A perpendicular to the plane of the triangle, we have



$$M\omega^2 \frac{a}{2\sqrt{3}} \tfrac{1}{2}a = Mg\,\frac{a}{2\sqrt{3}};$$

therefore $$\omega = \sqrt{\frac{2g}{a}}.$$

15819. (Professor Neuberg.)—a, b, c, R désignant les côtés d'un triangle ABC et le rayon du cercle circonscrit, trouver le maximum ou le minimum du rapport $(a^2+b^2+c^2) : R^2$.

Solution by Saradakanta Ganguli, M.A.; Professor Sanjána, M.A.; *and* C. M. Ross.

$$\frac{a^2+b^2+c^2}{R^2} = \frac{4R^2(\sin^2 A + \sin^2 B + \sin^2 C)}{R^2} = 4\,(\sin^2 A + \sin^2 B + \sin^2 C)$$

$$= u \text{ suppose.}$$

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Then $du = 0$; therefore $\sin 2A\,dA + \sin 2B\,dB + \sin 2C\,dC = 0$. Also $dA + dB + dC = 0$; therefore $\sin 2A = \sin 2B = \sin 2C$. Therefore either

$$A = B = C = 60° \quad \text{.....(1),}$$

or any two of A, B, C are each equal to 0° and the third = 180° ...(2). (1) gives the maximum value, viz. 9; and (2) the minimum value, viz. 0.

15806. (Rev. F. H. Jackson.)—Given

$$P_n(x) = \frac{[2n]!}{(2)_n [n]! [n]!} \left\{ x^n - p^3 \frac{[n][n-1]}{[2][2n-1]} x^{n-2} + \dots \right\},$$

in which the p prefixes of the terms of the series form the sequence

$$(p^3, p^8, p^{15}, \dots, p^{r(r+2)}, \dots),$$

and $[n] = (p^n - 1)/(p-1)$, $(2)_n = (1+p^n)(2)_{n-1}$, $[n]! = [n][n-1]!$, show that $\frac{[n]}{[2n-1]} \frac{P_n}{P_{n-1}}$ may be expressed as a continued fraction

$$\cfrac{}{x - \cfrac{\frac{[n]^2}{[2n+1][2n-1]} p^{n+2}}{x - \cfrac{\frac{[n+1]^2}{[2n+3][2n+1]} p^{n+3}}{x - \dots}}}$$

Solution by the Proposer.

It is easy to establish for P_n a recurrence form

$$[n] P_n = [2n-1] x P_{n-1} - p^{n+1} [n-1] P_{n-2},$$

reducing, in case $p = 1$, to the recurrence form of Legendre's function P_n; hence

$$\frac{[n+1] P_{n+1}}{[2n+1] P_n} = x - p^{n+2} \frac{[n] P_{n-1}}{[2n+1] P_n}$$

and

$$f_n \frac{[2n+1][2n-1]}{[n]^2} p^{-n-2} = \frac{1}{x - f_{n+1}} \quad \text{.....(A),}$$

where

$$f_n = \frac{[n] P_n}{[2n-1] P_{n-1}}.$$

From (A) the form of the continued fraction at once follows.

15794. (Robert W. D. Christie.)—Where p_n, q_n are any convergents of $p_n^2 - 2q_n^2 = \pm 1$, prove

$$2 \tan^{-1} \frac{q_n}{q_{n+1}} \pm \tan^{-1} \frac{1}{p_{2n+1}} = \tan^{-1} 1 = \tfrac{1}{4}\pi \quad \text{..... (1),}$$

$$2 \tan^{-1} \frac{p_n}{p_{n+1}} \mp \tan^{-1} \frac{1}{p_{2n+1}} = \tan^{-1} 1 = \tfrac{1}{4}\pi \quad \text{..... (2);}$$

and show that Euler's and Machin's series are simple cases (signs alternate).

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Solution by the PROPOSER.

A few preliminary results are desirable.

(*a*) $p_{n+1} = q_{n+1}+q_n, \quad p_n = q_{n+1}-q_n$ ..... (1, 2);

for $p^2_{n+1}-2q^2_{n+1} = \pm 1$ and $p^2_n-2q^2_n = \mp 1$ from the general equation. Adding and equating, $p^2_{n+1}+p^2_n = 2(q^2_{n+1}+q^2_n) = (q_{n+1}+q_n)^2+(q_{n+1}-q_n)^2$; thus $p_{n+1} = q_{n+1}+q_n, \quad p_n = q_{n+1}-q_n$.

(*b*) From (1), (2) by addition and subtraction,

$p_{n+1}+p_n = 2q_{n+1}, \quad p_{n+1}-p_n = 2q_n$ ..... (3, 4).

(*c*) We have, by *Reprint*, Vol. VI., p. 99,

$$p^2_{2n+1}+1 = (p_{n+1}q_{n+1}+p_nq_n)^2+(p_{n+1}q_n-p_nq_{n+1})^2;$$

thus $p_{2n+1} = p_{n+1}q_{n+1}+p_nq_n, \quad \pm 1 = p_{n+1}q_n-p_nq_{n+1}$.....(5, 6).

(*d*) $2q_nq_{n+1} = q^2_{n+1}-q^2_n \pm 1$, for $p^2_{n+1}-2q^2_{n+1} = \pm 1$ and $p_{n+1} = q_{n+1}+q_n$; therefore $2q_nq_{n+1} = q^2_{n+1}-q^2_n \pm 1$.

A. We have to show that $2\tan^{-1}\frac{p_n}{p_{n+1}} \mp \tan^{-1}\frac{1}{p_{2n+1}} = \tan^{-1}1 = \frac{1}{4}\pi$.

$$2\tan^{-1}\frac{p_n}{p_{n+1}} = \frac{2p_np_{n+1}}{p^2_{n+1}-p^2_n} = \frac{2p_np_{n+1}}{2q_{n+1}+2q_n} = \frac{q^2_{n+1}-q^2_n}{2q_nq_{n+1}} = \frac{q^2_{n+1}-q^2_n}{q^2_{n+1}-q^2_n \pm 1};$$

also $p_{2n+1} = p_{n+1}q_{n+1}+p_nq_n$ where $p_{n+1} = q_{n+1}+q_n$ and $p_n = q_{n+1}-q_n$; thus $p_{2n+1} = q^2_{n+1}-q^2_n+2q_nq_{n+1} = 2q^2_{n+1}-2q^2_n \pm 1$.

Now, for brevity, let $a = q^2_{n+1}-q^2_n$; thus the theorem to be proved is practically then $p^{2n+1} = 2a+1 = 2q^2_{n+1}-2q^2_n+1$,

$$\tan^{-1}\frac{a}{a+1}+\tan^{-1}\frac{1}{2a+1} = \frac{\frac{a}{a+1}+\frac{1}{2a+1}}{1-\frac{a}{(a+1)(2a+1)}} = \tan^{-1}1 = \tfrac{1}{4}\pi.$$

Again, let $a = q^2_{n+1}-q^2_n-1$; then $p_{2n+1} = 2a+1 = 2q^2_{n+1}-2q^2_n-1$; thus

$$\tan^{-1}\frac{a+1}{a}-\tan^{-1}\frac{1}{2a+1} = \frac{\frac{a+1}{a}-\frac{1}{2a+1}}{1+\frac{a+1}{a(2a+1)}} = \tan^{-1}1 = \tfrac{1}{4}\pi.$$

B. $$2\tan^{-1}\frac{q_n}{q_{n+1}} \pm \tan^{-1}\frac{1}{p_{2n+1}} = \tan^{-1} = \tfrac{1}{4}\pi,$$

for $$2\tan^{-1}\frac{q_n}{q_{n+1}} = \frac{2q_nq_{n+1}}{q^2_{n+1}-q^2_n} = \frac{q^2_{n+1}-q^2_n \pm 1}{q^2_{n+1}-q^2_n}.$$

Let $a = q^2_{n+1}-q^2_n+1$; thus $2a-1 = 2q^2_{n+1}-2q^2_n+1$; therefore

$$\tan^{-1}\frac{a}{a-1}-\tan^{-1}\frac{1}{2a-1} = \frac{\frac{a}{a-1}-\frac{1}{2a-1}}{1+\frac{a}{(a-1)(2a-1)}} = \tan^{-1}1 = \tfrac{1}{4}\pi;$$

and again, $a = q^2_{n+1}-q^2_n-1$; then $2a+1 = 2q^2_{n+1}2q^2_n-1$; therefore

$$\tan^{-1}\frac{a}{a+1}+\tan^{-1}\frac{1}{2a+1},$$

as before.

C. It follows, since A and B each equal unity, that

$$\tan^{-1}\frac{q_n}{q_{n+1}}+\tan^{-1}\frac{p_n}{p_{n+1}} = \tan^{-1}1,$$

which is proved directly from the results given.

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2357. (T. Cotterill, M.A.)—(1) If the determinant $\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}$ vanish, the equations

$$\alpha(bz^2+cy^2-2dyz) = \beta(cx^2+az^2-2ezx) = \gamma(ay^2+bx^2-2fxy)$$

are satisfied by two conjugate pairs of values of the variables, one pair being independent of the constants α, β, γ. Find the equation to the quadric satisfying the values $y = z = 0$, $z = x = 0$, $x = y = 0$, and the last pair of values, and the linear equations satisfying each conjugate pair. (2) Show that such a system of equations is the analytical representation of the projection of the three circles described on the lines connecting the opposite points of intersection of four lines in a plane as diameters, the circle circumscribing their diagonal triangle, and the line at infinity.

Solution by Professor Nanson.

The locus of a point such that the tangents from it to two conics given tangentially by $\Sigma = 0$, $\Sigma' = 0$ form a harmonic pencil is a third conic given punctually by $S = 0$, where the coefficients of S are linear in the coefficients of Σ and in those of Σ'. Hence, if Σ' belongs to a range, then S belongs to a pencil. Taking the four common tangents of the range to be $a\xi^2 = \beta\eta^2 = \gamma\zeta^2$, we have

$$\Sigma = (abcfgh)(\xi\eta\zeta)^2, \quad \Sigma' = a'\xi^2+b'\eta^2+c'\zeta^2, \quad S = a'A'+b'B'+c'C',$$

where $A' = bz^2+cy^2-2fyz$, ..., and $a'/a+b'/\beta+c'/\gamma = 0$.

The four vertices of the pencil are seen, by taking a', b', c' in turn to be zero, to be given by $\alpha A' = \beta B' = \gamma C'$ .....(1),

and these are the equations given in the Question, d, e, f having been replaced by f, g, h. Now, if the discriminant Δ of Σ vanishes, the three equations $A' = 0$, $B' = 0$, $C' = 0$ have two common solutions, viz., the co-ordinates of the two points now given by $\Sigma = 0$. These two points—P, Q, say—are independent of $\alpha : \beta : \gamma$ and are two of the four vertices of the pencil.

Now the co-ordinates of P, Q are given by

$$\xi^2/A = \eta^2/B = \zeta^2/C = \eta\zeta/F = \zeta\xi/G = \xi\eta/H,$$

where A, ... are the co-factors of a, ... in Δ. Hence the equation of PQ is $x/F+y/G+z/H = 0$ ... .(2),

and this is one of the linear equations asked for. By combining this linear equation with any one of the quadratic equations $A' = 0$, $B' = 0$, $C' = 0$, we readily find the quadric relation asked for to be

$$ayz/F+bzx/G+cxy/H = 0 \text{.....(3).}$$

One of the conics of the range touches PQ, viz., the one determined by

$$Aa'+Bb'+Cc' = 0,$$

and the corresponding S conic is

$$(1/\alpha)(BC'-B'C)+(1/\beta)(CA'-C'A)+(1/\gamma)(AB'-A'B) = 0.$$

It is obvious from geometry that this conic breaks up into the line PQ and the join of the remaining two vertices of the pencil. Accordingly,

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we readily verify that the last equation breaks up, in virtue of the condition $\Delta = 0$, into (2) and

$$\frac{1}{\alpha}\left(\frac{gx-as}{H}-\frac{hx-ay}{G}\right)+\frac{1}{\beta}\left(\frac{hy-bx}{F}-\frac{fy-bs}{H}\right)+\frac{1}{\gamma}\left(\frac{fs-cy}{G}-\frac{gz-cx}{F}\right)=0 \quad \ldots\ldots\ldots\ldots(4).$$

This is the second linear relation asked for.

The three conics (1) are conics of the pencil, and the corresponding conics of the range are flat conics coinciding with the diagonals of the quadrilateral formed by the four common tangents. Each of the three conics (1) is therefore the locus of a point at which PQ and one of the three diagonals subtend a harmonic pencil. When then P, Q are at the focoids the conics (1) are the circles on the diagonals, their finite intersections are the two points defined by Clifford as the foci of the quadrilateral; (2) is the line at infinity; (3) is the circum-circle of the diagonal triangle; and (4) is the directrix of the parabola of the range.

15772. (R. W. D. Christie.)—Prove $A^2 = B^2 + C^2$ (integers) when $A = \Sigma_0^n \{(m+n)!/[2n!\,(m-n)!]\}$ $(m > n)$.

E.g., $m = 5$, $n = 5, 4, 3, 2, 1, 0$; then

$$A = \Sigma_0^5 (1 + 9 + 28 + 35 + 15 + 1) = 8^2 + 5^2.$$

Solution by F. N. Mayers, M.A.

If we indicate the proposed expression by u_{2n-1}, and the parallel expression $\Sigma_1^{n+1} \{(m+n)!/[(2n-1)!\,(m-n+1)!]\}$ by u_{2n}, then u_l means the number of ways in which l playing cards can be dealt out without any succession of two or more red occurring if each card is returned to the pack every time. It can be immediately inferred from this that u_l obeys the law $u_{l+1} = u_l + u_{l-1}$ (Art. "Probability," *Encyclopædia Britannica*, Section 1). But it can also be proved from the expressions themselves. The manner of addition is shown as follows:—

$$\begin{aligned} u_5 &= \quad 1 + 5 + 6 + 1 \\ u_6 &= 1 + 6 + 10 + 4 \\ \hline u_7 &= 1 + 7 + 15 + 10 + 1 \\ u_6 &= \quad 1 + 6 + 10 + 4 \\ \hline u_8 &= 1 + 8 + 21 + 20 + 5 \end{aligned}$$

It appears, therefore, that the r-th term of u_l, added to the $(r+1)$-th term of u_{l+1}, is equal to the $(r+1)$-th term of u_{l+2}; and this can be shown to be always the case. It will be seen that u_{2n} has the same number of terms as u_{2n-1}; hence, in this manner of addition, in adding u_{2n-1} to u_{2n}, there is unity overlapping one each side, and u_{2n+1} begins and ends in 1, as it should do; but, in adding u_{2n-2} to u_{2n-1}, u_{2n-1} has one more term, and therefore no term overlaps at the end; so that u_{2n} begins with 1 and ends with $n+1$, as it should do.

To prove the r-th term of u_l + $(r+1)$-th term of u_{l+1} = $(r+1)$-th term of u_{l+2}. The $r+1$-th term of u_{2n-1} is $(2m-r)!\,[(2m-2r)!\,r!]$. The

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$(r+2)$-th term of u_{2m} is $(2m-r)!/[(2m-2r-1)!\,(r+1)!]$. Their sum is

$$(2m-r)!/[(2m-2r)!\,(r+1)!]\,(r+1+2m-2r)$$
$$= (2m-r+1)!/[(2m-2r)!\,(r+1)!] = (r+2)\text{-th term of } u_{2m+1},$$

i.e., $u_{2m+1} = u_{2m} + u_{2m-1}$. The $(r+1)$-th term of u_{2m} is

$$(2m-r+1)!/[(2m-2r+1)!\,r!];$$

the difference between this and the $(r+1)$-th term of u_{2m-1} is

$$(2m-r)!/[(2m-2r+1)!\,r!]\,[2m-r+1-(2m-2r+1)]$$
$$= (2m-r)!/[(2m-2r+1)!\,(r-1)!] = r\text{-th term of } u_{2m-2},$$

i.e., $u_{2m} = u_{2m-1} + u_{2m-2}$. Hence the relationship in question is proved, whether l is even or odd. The equation $u_{n+1} = u_n + u_{n-1}$ gives $E^2-E-1 = 0$; and, since $u_1 = 2$, $u_2 = 3$, we get, solving for E,

$$u_n = (\alpha^{n+2} - \beta^{n+2})/(\alpha-\beta),$$

where α, β are roots of $x^2-x-1 = 0$. Then

$$\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)^2 + \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2$$
$$= \frac{\alpha^{2(n+2)} + \beta^{2(n+2)} \pm 2 + \alpha^{2(n-1)} + \beta^{2(n-1)} \mp 2}{(\alpha-\beta)^2} \quad (\text{since } \alpha\beta = -1)$$
$$= \frac{\alpha^{2n+4} + \beta^{2n+4} + \alpha^{2n+2} + \beta^{2n+2}}{(\alpha-\beta)^2}$$
$$= \frac{\alpha^{2n+4} - \beta\alpha^{2n+3} + \beta^{2n+4} - \alpha\beta^{2n+3}}{(\alpha-\beta)^2} \quad (\text{since } \alpha\beta = -1)$$
$$= \frac{(\alpha-\beta)(\alpha^{2n+3} - \beta^{2n+3})}{(\alpha-\beta)^2} = u^{2n+1};$$

therefore
$$u_{2n+1} = u_n^2 + u_{n-1}^2.$$

This proves the required relationship. It might, of course, be also deduced from the fact that the series 2, 3, 5, ... form numerators (or denominators) to successive convergents of the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}.$$

It would, doubtless, be more elegant to prove the relationship

$$u_{2n+1} = u_n^2 + u_{n-1}^2$$

direct from the expression itself, without the intermediate relation

$$u_{n+1} = u_n + u_{n-1}.$$

The PROPOSER contributes the following:—

The general solution of $(a+y)(y_{n+1}^2 - az_{n+1}^2) = 4$, where a is arbitrary, is

$$y_{n+1} = 1,\ a+1,\ a^2+3a+1,\ a^3+5a^2+6a+1,\ \ldots\ (a+2)\,q_{n+1} - q_n \quad \ldots \quad (\mathrm{A}),$$

with a corresponding value for z_{n+1} [see this Question (15716) in a previous volume), or

$$y_{n+1} = \sum_0^n \left\{ \frac{(m+n)!}{(2n)!\,(m-n)!}\, a^n \right\} \quad (\mathrm{A}),$$

and a corresponding value for z_{n+1}.

Let $a = 1$; then

$$y_{n+1} = 1,\ 2,\ 5,\ 13,\ 34,\ 89,\ 233,\ 610,\ \ldots\ 3q_n - q_{n-1} \quad (\mathrm{A})$$

and
$$5y_{n+1}^2 - z_{n+1}^2 = 4,$$

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Consequently $y_{n+1} = \sum_{0}^{n} \left\{ \frac{(m+n)!}{(2n)!\,(m-n)!} \right\}$;

for, if $C_{m+1} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610$, &c.,
the continuant series ..... (C),

we have $y_{n+1} = C_{2m+1} = C_{2m} + C_{2m-1} = C_m^2 + C_{m-1}^2$ ..... (D)
by established results.

Also it is easily shown that

$C_{m+3} = 1 + \sum_{1}^{m+1} C_{n+1}$ or $C_{2m+1} = 1 + \sum_{1}^{2m-1} C_{2n-1} = C_m^2 + C_{m-1}^2$ from (D).

E.g., let $m = 7$, $n = 7, 6, 5, 4, 3, 2, 1, 0$; then

$$y_8 = C_{15} = 610 = C_7^2 + C_6^2 = 21^2 + 13^2 = 1 + \sum_{1}^{13} C_{13} = \sum_{0}^{n} \left\{ \frac{(m+n)!}{(2n)!\,(m-n)!} \right\}$$

$$= \left\{ \frac{(7+7)!}{(14)!+0!} + \frac{(7+6)!}{(12)!+1!} + \frac{(7+5)!}{(10)!+2!} + \frac{(7+4)!}{8!+3!} + \frac{(7+3)!}{6!+4!} + \frac{(7+2)!}{4!+5!} + \frac{(7+1)!}{2!+6!} + \frac{(7+0)!}{0!+7!} \right\}$$

$$= 1 + 13 + 66 + 165 + 210 + 126 + 28 + 1.$$

The PROPOSER also remarks that:—"The relation desiderated by Mr. Mayers was proved by Professor Sircom and others, and appears in an earlier volume of the *Mathematical Reprint*."

15781. (Professor NEUBERG.)—Soit A le sommet d'une parabole. En un point quelconque M de cette courbe on mène une corde MN perpendiculaire à MA. Trouver le lieu du milieu de cette corde.

Solutions (I.) *by* J. A. H. JOHNSTON, M.A., *and* Professor SANJÁNA, M.A.; (II.) *by* W. AUSTIN SLEIGH, B.A., *and* L. ISSERLIS, B.A.; (III.) *by* A. M. NESBITT, M.A., *and* F. W. REEVES, B.A.

(I.) Let the chord be given by $x'y'$ and $x''y''$, and be at right angles to the line from 0 to $x'y'$. If the middle point be (ξ, η), and $y^2 = 4ax$ the parabola, $2\xi = x' + x''$ or $8a\xi = y'^2 + y''^2$, $2\eta = y' + y''$ .....(1). The direction of the chord is given by $\tan\theta = 4a/(y' + y'') = 2a/\eta$, where $y'/x' = -\eta/2a$, and therefore $y' = -8a^2/\eta$ from (1); therefore

$$y''^2 = 8a\xi - 64a^4/\eta^2 \quad \text{and} \quad y'' = (2/\eta)(\eta^2 + 4a^2);$$

therefore $(\eta^2 + 4a^2)^2 = 2a\xi\eta^2 - 16a^4$,

or the locus is $(y^2 + 4a^2)^2 = 2axy^2 - 16a^4$.

(II.) The equation to the chord MN, mid-point P (h, k) of the parabola $y^2 = 4ax$, is (Loney, § 221, p. 195)

$$k(y-k) = 2a(x-h).$$

This passes through the point M $(at^2, 2at)$ (Loney's notation, p. 199); therefore $k(2at-k) = 2a(at^2-h)$ ..... (1).

Now the equation to AM is $y = (2/t)x$;

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therefore, since MN and AM are perpendicular, $(2/t)(2a/k) = -1$, *i.e.*,

$$t = -4a/k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substituting for t from (2) in (1), we obtain on reduction the required locus of P, viz., $(y^2+4a^2)^2-2axy^2+16a^4 = 0$.

(III.) Take $y^2 = px$ for the parabola, and let the co-ordinates of M be pm^2, pm. The equation to MN is therefore $y-pm = -(x-pm^2)\,m$, which cuts the curve again at the point given by $p = -m\,(y+pm)$, whose co-ordinates are thus $y = -[p\,(1+m^2)]/m$, $x = p\,[(1+m^2)/m]^2$. Hence, if (α, β) be the mid-point of MN,

$$2\alpha/p = m^2+[(1+m^2)/m]^2, \quad 2\beta/p = -1/m,$$

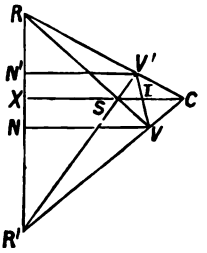
whose locus is the quartic $8\beta^4+4p^2\beta^2+p^4 = 4\alpha\beta^2p$; or, if $q = 2p$,

$$\beta^4+2q^2\beta^2+2q^4 = q\alpha\beta^2.$$

15779. (Professor SANJÁNA, M.A.)—Prove, by the elementary geometry of the conic sections, that the sum of the reciprocals of two perpendicular focal chords of a conic is of constant magnitude.

Solution by Professor NANSON.

Let the perpendicular focal chords PQ, P′Q′ cut the directrix in R, R′; then their middle points V, V′ lie on CR′, CR. Now, if VN, V′N′ are the perpendiculars from V, V′ on the directrix, we have $PQ = 2e\,NV$. Hence we have to prove the harmonic mean of NV, N′V′ to be constant. But, if VV′ cuts SC in I, then XI is the harmonic mean of NV, N′V′ and also of XS, XC. Therefore, &c.



15758. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Show that the product of the two series

$$1^3+2^3+3^3+4^3+5^3+\ldots+n^3, \quad 1^5+2^5+3^5+4^5+5^5+\ldots+n^5$$

may be represented as a single series.

Solution by C. M. Ross *and* KELA DEVA RAU, B.A.

$$1^3+2^3+3^3+\ldots+n^3 = \tfrac{1}{4}n^2\,(n+1)^2,$$
$$1^5+2^5+3^5+\ldots+n^5 = \tfrac{1}{12}n^2\,(n+1)^2\,(2n^2+2n-1);$$

therefore the product of the series is $\frac{1}{48}n^4\,(n+1)^4\,(2n^2+2n-1)$, which may be taken as the sum of n terms of a certain series. Hence the sum of

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$n-1$ terms of this certain series is obtained by putting $n-1$ for n; therefore $S_{n-1} = \frac{1}{48}n^4(n-1)^4(2n^2-2n-1)$.

Therefore the n-th term of the series

$= S_n - S_{n-1} = \frac{1}{48}n^4[(2n^2-1)\{(n+1)^4-(n-1)^4\} + 2n[(n+1)^4+(n-1)^4]$

$= \frac{1}{12}n^5(5n^4+8n^2-1)$.

Thus the required single series is

$$1+296+9639+\ldots+\tfrac{1}{12}n^5(5n^4+8n^2-1).$$

2439. (T. Cotterill, M.A.)—If a, b, c, d are collinear points as well as x, y, z, t, and in the same plane, then, of the sixteen intersections of ax, ay, bx, by with cz, ct, dz, dt, eight lie on one conic and eight on another. The four points of intersection of these conics lie on a third conic through a, b, x, y and a fourth through c, d, z, t, and these conics are respectively harmonics to the lengths (ab, xy) and (cd, zt). Also the tangents to the four conics at any point of intersection are harmonic.

Solution by Professor Nanson.

Take the bases of the ranges as axes of x, y; denote the eight points by $a, b, c, d, a', b', c', d'$ and the reciprocals of their distances from the origin by $a, b, c, d, a', b', c', d'$, and let $(aa') = ax+a'y-1$; then the equation

$$S_1 \equiv \frac{(aa')(bb')}{(a-b)(a'-b')} - \frac{(cc')(dd')}{(c-d)(c'-d')} = 0$$

clearly represents a conic through the four meets

$$aa', cc'; \quad aa', dd'; \quad bb', cc'; \quad bb', dd' \quad \ldots\ldots\ldots\ldots (1).$$

But S_1 also passes through the four meets

$$ab', cd'; \quad ab', dc'; \quad ba', cd'; \quad ba', dc' \quad \ldots\ldots\ldots\ldots (2).$$

For at the first of these meets we have $ax+b'y-1=0$, $cx+d'y-1=0$, and therefore at this point S_1 reduces to

$$\frac{(a'-b')y(b-a)x}{(a-b)(a'-b')} - \frac{(c'-d')x(d-c)y}{(c-d)(c'-d')},$$

i.e., to zero. Also S_1 is not altered by interchanging c with d and c' with d', or by interchanging a with b and a' with b'. Hence S_1 passes through the eight meets (1), (2). It is obvious geometrically, and may be verified algebraically, that S_1 can be put in the form

$$S_1 \equiv \frac{(ab')(ba')}{(a-b)(a'-b')} - \frac{(cd')(dc')}{(c-d)(c'-d')}.$$

Similarly, the other eight meets lie on a conic S_2 where

$$S_2 \equiv \frac{(aa')(bb')}{(a-b)(a'-b')} + \frac{(cd')(dc')}{(c-d)(c'-d')} \equiv \frac{(ab')(ba')}{(a-b)(a'-b')} + \frac{(cc')(dd')}{(c-d)(c'-d')}.$$

Now we have identically

$$(a-b)(a'-b')(S_1+S_2) = (aa')(bb') + (ab')(ba') \equiv S_3,$$
$$(c-d)(c'-d')(S_2-S_1) = (cc')(dd') + (cd')(dc') \equiv S_4.$$

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Thus the four meets of S_1, S_2 lie on S_3 and S_4, which go through $aba'b'$, $cdc'd'$ respectively. Again, since S_3 is not altered when a, b are interchanged, S_3 is harmonic to the lengths ab, $a'b'$. Also, since S_3, S_4 are of the forms $\lambda S_1 + \lambda S_2$, $\mu S_1 - \mu S_2$, the tangents at any meet of the four form a harmonic pencil.

15804. (Robert W. D. Christie.) — Let $p = m^4 - m^3 + m^2 - m + 1$. Prove $[(m^2+1)(m-1)]^{n+1} + m^{4n-1} = 0 \bmod p$ and generalize.

Solution by Lt.-Col. Allan Cunningham, R.E., *and* Alexander Holm, M.A.

Here $m^4 \equiv m^3 - m^2 + m - 1 = (m^2+1)(m-1) \pmod p$; therefore

$$[(m^2+1)(m-1)]^{n+1} + m^{4n-1} \equiv (m^4)^{n+1} + m^{4n-1} \pmod p$$
$$\equiv m^{4n-1}(m^5+1) \pmod p \equiv 0 \pmod p,$$

for $p = (m^5+1)/(m+1)$. To generalize this, let $p = (m^r+1)/(m+1)$ (r an odd number); therefore $p = [m^{r-1} - \mathbf{M}]$ where

$$\mathbf{M} = m^{r-2} - m^{r-3} + \ldots + m - 1 = [(m^{r-3} + m^{r-5} + \ldots + m^2 + 1)(m-1)].$$

(Here $\mathbf{M}$ corresponds to the quantity in the square brackets above. Hence $\mathbf{M} \equiv m^{r-1} \pmod p$; therefore

$$\mathbf{M}^{n+1} + m^{(r-1)n-1} \equiv m^{(r-1)(n+1)} + m^{(r-1)n-1} \pmod p \equiv m^{(r-1)n-1}(m^r+1)$$
$$\equiv 0 \pmod p.$$

This is a strict generalization of the above theorem. A similar result can be obtained by taking $p = (m^r-1)/(m-1)$ [where r may be either odd or even].

The Proposer's solution is as follows:—Let

$$m^\nu_{+\sigma} = m^\nu + m^{\nu-1} + m^{\nu-2} + \ldots + 1, \quad m^\nu_{-\sigma} = m^\nu - m^{\nu-1} + m^{\nu-2} - \ldots + 1.$$

We have then $(m+1)(m^\nu_{-\sigma}) = m^{\nu+1} + 1 = 0 \pmod{}^\nu_{-\sigma}$; therefore

$$m^{n\nu-1}(m^{\nu+1}+1) = 0 \pmod p;$$

therefore $m^{\nu(n+1)} + m^{\nu n-1} = 0 \pmod p$. But $m^\nu = [m^{\frac{1}{2}\nu} + (-1^{\frac{1}{2}\nu})](m^{\frac{1}{2}(\nu-2)}_{-\sigma})$; thus $\{[m^{\frac{1}{2}\nu} + (-1^{\frac{1}{2}\nu})][m^{\frac{1}{2}(\nu-2)}_{-\sigma}]\}^{n+1} + m^{\nu n-1} = 0 \pmod p$. In the given example let $\nu = 4$. Then

$$[(m^2+1)(m-1)]^{n+1} + m^{4n-1} = 0 \pmod{m^4 - m^3 + m^2 - m + 1} \quad \ldots\ldots (1).$$

Consequently, if $n = -n$, we get

$$[(m^2+1)(m-1)]^{n-1} + m^{4n+1} = 0 \pmod{m^4 - m^3 + m^2 - m + 1} \quad \ldots\ldots (2),$$

and, if $m = -m$, then we can secure an equation for

$$\pmod{m^\nu_{+\sigma}} \quad \ldots\ldots (3),$$

&c., &c.

15782. (A. M. Nesbitt, M.A.)—The inscribed circle of a triangle ABC touches the sides AC, AB in E and F, and an escribed circle touches the same sides produced in E′ and F′. If the median AK cut EF in G and E′F′ in G′, prove that BGCG′ is a parallelogram.

[N.B.—It is only possible to publish a few of the numerous Solutions sent in.—Ed.]

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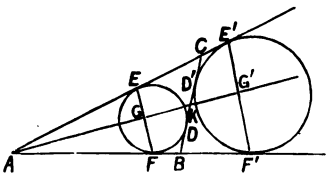
Solutions (I.) *by* W. J. Dobbs, M.A.; (II.) *by* B. C. Wallis, F.C.P., B.Sc. (Econ.); (III.) *by* R. J. Whitaker.

(I.) Since $BD = s-b = CD'$, the middle point K of BC is also the middle point of DD'. Now the radical axis of the two circles bisects their common tangents DD', EE', FF' and is parallel to EF and E'F'. Therefore K is midway between the parallels EF and E'F'. Thus GG' and BC bisect each other at K and are the diagonals of a parallelogram.



(II.) $EE' = FF' = a$; $AE' = AF' = s$;

$$AE = AF = (s-a);$$

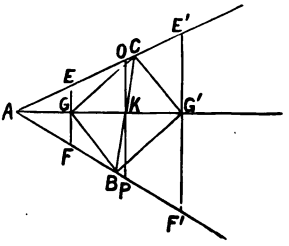
$$BF' = s-c = \tfrac{1}{2}(a+b+c)-c = \tfrac{1}{2}(a+b-c);$$

$$EC = b-(s-a)$$

$$= b+a-\tfrac{1}{2}(a+b+c) = \tfrac{1}{2}(a+b-c);$$

therefore $BF' = EC$ and $BF = CE'$.

Through K draw OKP parallel to EF; then, since $BF = CE'$, ..., O is the middle point of EE', and P is the middle point of FF'. Therefore K is the middle point of GG'. Since BC and GG' bisect each other, BGCG' is a parallelogram.



(III.) Draw CL, KM parallel to EF. Now, if $2s = a+b+c$, we have $AF = s-a$ and $AF' = s$; therefore $FF' = a$. Also $AL = AC$ (for $AF = AE$ and CL is parallel to FE); therefore

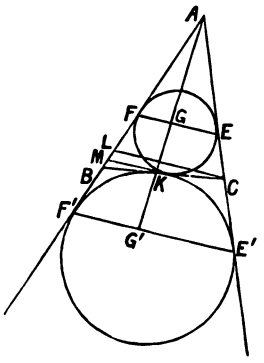
$$FL = b-(s-a) = a+b-s = 2s-c-s$$

$$= s-c = F'B.$$

Also $LM = BM$ (for K is the middle point of BC); therefore

$$FL+LM = F'B+BM,$$

i.e., M is the middle point of FF'. But MK, FG, F'G' are all parallel; therefore K is the middle point of GG'; therefore GG', BC bisect one another; whence BGCG' is a parallelogram.



15755. (Professor Sanjána, M.A.)—Prove that the expansion of $\operatorname{sech}^{-1} x^{\frac{1}{2}} + \log x^{\frac{1}{2}}$ is

$$\log 2 - \left(\frac{1}{2}\,\frac{x}{2} + \frac{1.3}{2.4}\,\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\,\frac{x^3}{6} + \ldots\right),$$

x lying between zero and unity.

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Solution by C. M. Rosa.

Let $\alpha = \operatorname{sech}^{-1} x^{\frac{1}{2}}$; therefore $\operatorname{sech} \alpha = x^{\frac{1}{2}}$; therefore $2/(\epsilon^{\alpha} + \epsilon^{-\alpha}) = x^{\frac{1}{2}}$; therefore $x^{\frac{1}{2}}\epsilon^{2\alpha} - 2\epsilon^{\alpha} + x^{\frac{1}{2}} = 0$; therefore $\epsilon^{\alpha} = [1 \pm (1-x)^{\frac{1}{2}}]/x^{\frac{1}{2}}$; therefore $\alpha = \log[1 + (1-x)^{\frac{1}{2}}] - \log x^{\frac{1}{2}}$ (positive sign being taken); therefore

$$\begin{aligned}\operatorname{sech}^{-1} x^{\frac{1}{2}} + \log x^{\frac{1}{2}} &= \log[1 + (1-x)^{\frac{1}{2}}] = \log[1 + 1 - \tfrac{1}{2}x - (\tfrac{1}{4}/2!)\,x^2 - \ldots] \\ &= \log\left\{2[1 - (\tfrac{1}{4}x + \tfrac{1}{16}x^2 + \tfrac{1}{32}x^3 + \ldots)]\right\} \\ &= \log 2 + \log[1 - (\tfrac{1}{4}x + \tfrac{1}{16}x^2 + \tfrac{1}{32}x^3 + \ldots)] \\ &= \log 2 - \left(\frac{1}{2}\,\frac{x}{2} + \frac{1.3}{2.4}\,\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\,\frac{x^3}{6} + \ldots\right).\end{aligned}$$

15733. (Professor M. W. Crofton, F.R.S.)—An urn contains two balls, A and B. The probability that A is black is p; that B is black, q. One ball is now drawn, and found to be black:—(1) What is the chance that this is A or B? (2) What is now the probability that A is black or B? (3) If the ball drawn is replaced in the urn, and a fresh drawing made, what is the chance it gives a black ball?

[The balls A, B may be supposed to have been taken from two bags containing black and white balls in given proportions.]

Solution by F. N. Mayers.

1. Adopting the usual notation where a known result has occurred, viz., w_n = probability of the existence of the n-th cause, p_n the probability that, if this cause exists, the given result will ensue, $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$, $p_1 = p$, $p_2 = q$. Hence the probability that the ball is A

$$= \tfrac{1}{2}p/(\tfrac{1}{2}p + \tfrac{1}{2}q) = p/(p+q);$$

and, similarly, the probability that it is B $= q/(p+q)$. The sum of these probabilities is equal to unity, as it should be, for it must be A or B.

2. The probabilities remain the same as before, with the exception that the possibility of both balls being white must now be omitted from the denominator, *i.e.*, the denominator, instead of being unity, is

$$1 - (1-p)(1-q) = p + q - pq,$$

and the required probabilities are $p/(p+q-pq)$, $q/(p+q-pq)$.

3. The other probabilities will be altered in the same ratio, *i.e.*, the probability that both balls are black will be $pq/(p+q-pq)$, and that one is black and the other white will be respectively

$$[p(1-q)]/(p+q-pq), \quad [q(1-p)]/(p+q-pq).$$

(This would be most easily seen by putting fractions for p, q, viz., l/m and t/s, where m and s are total numbers of balls from the bags from which the balls are taken, and l, t are the number of black balls in each bag, $m-l$ and $t-s$ being therefore, of course, the number of white.)

Hence the probability required is

$$pq/(p+q-pq) + [\tfrac{1}{2}p(1-q)]/(p+q-pq) + [\tfrac{1}{2}q(1-p)]/(p+q-pq).$$

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For, if both balls are black, it is certain that a black ball will be chosen, whilst, if one is white and the other black, it is an even chance.

Or, required probability

$$= [pq + \tfrac{1}{2}(p+q-pq)]/(p+q-pq) = (p+q+pq)/[2(p+q-pq)].$$

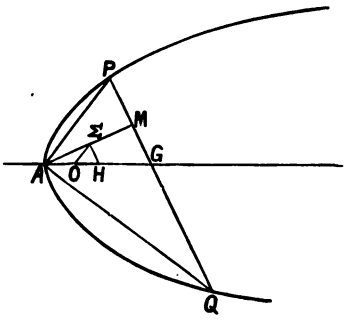
15746. (The late R. Tucker, M.A.)—PQ is a chord of an ellipse, subtending a right angle at the vertex. Find the locus of the S.-point of the triangle PAQ.

Solution by Professor Sanjâna, M.A., *and* W. F. Beard, M.A.

As the angle PAQ is a right angle, PQ goes through a fixed point on the normal at A, *i.e.*, on the axis; this point G is such that

$$AG = 2ab^2/(a^2 \pm b^2)$$

for central conics.



Draw AM perpendicular to PQ and bisect it in Σ, which is therefore the symmedian point of PAQ. Bisect AG in H and join HΣ; then HΣA is a right angle. Hence Σ lies on the circle whose centre O is the mid-point of AH, and whose radius OΣ is half of AH, *i.e.*, of length $ab^2/2(a^2 \pm b^2)$.

10596. (Professor Déprez.) — Soient a, b, c les côtés d'un triangle sphérique ABC; r le rayon du cercle inscrit; x, y, z les distances d'un point de ce cercle aux côtés de ABC. Démontrer la formule

$$\cos \tfrac{1}{2}A (\sin y \sin z)^{\frac{1}{2}} + \ldots = 2 \tan r \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

Solution by E. W. Middlemast.

Taking OA, OB, OC as oblique axes, let λ, μ, ν be the direction cosines of the point P. Let n have its usual meaning for the triangle ABC and let n_1, n_2, n_3 have similar meanings for BPC, CPA, APB, so that $2n_1 = \sin x \sin a$, ... Now, projecting OP on the perpendicular to the plane OBC, $\sin x = \lambda \sin AD$, where AD is perpendicular to BC; whence $n_1 = n\lambda$, ... Hence

$$n^2 = \Sigma n_1^2 + 2\Sigma n_2 n_3 \cos a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If λ', μ', ν' be the direction cosines of I the in-centre, $\lambda' = (\sin a \sin r)/2n$, ... Hence

$$\cos r = \cos IP = \Sigma\lambda\lambda' + \Sigma(\mu\nu' + \mu'\nu)\cos a = \Sigma\lambda(\lambda' + \mu' \cos b + \nu' \cos c)$$
$$= (\sin r/2n^2)\Sigma n_1[\sin a + \sin(b+c)] = (\sin r/n^2)\Sigma n_1 \sin s \cos(s-a),$$

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and, since $\tan r = n/\sin s$,

$$\Sigma n_1 \cos(s-a) = n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) it follows that

$$\Sigma n_1^2 \sin^2(s-a) - 2\Sigma n_2 n_3 [\cos a - \cos(s-a)\cos(s-b)] = 0$$

or $$\Sigma n_1^2 \sin^2(s-a) - 2\Sigma n_2 n_3 \sin(s-b)\sin(s-c) = 0$$

or $$\Sigma \surd[n_1 \sin(s-a)] = 0;$$

therefore $$-n_1 = \surd\left[n_1 n_2 \frac{\sin(s-b)}{\sin(s-a)}\right] + \surd\left[n_3 n_1 \frac{\sin(s-c)}{\sin(s-a)}\right], \ \ldots;$$

therefore

$$-n = -\Sigma n_1 \cos(s-a)$$

$$= \Sigma \surd(n_2 n_3) \left\{\sqrt{\left[\frac{\sin(s-b)}{\sin(s-c)}\right]}\cos(s-c) + \sqrt{\left[\frac{\sin(s-c)}{\sin(s-b)}\right]}\cos(s-b)\right\}$$

$$= \Sigma \surd(n_2 n_3) \frac{\sin a}{\surd[(\sin(s-b)\sin(s-c)]} = \Sigma \frac{\surd(n_2 n_3)\sin a}{\sin \frac{1}{2}A \surd(\sin b \sin c)}$$

$$= \tfrac{1}{2}\Sigma \surd(\sin y \sin z) \frac{\sin a}{\sin \frac{1}{2}A} = \Sigma \surd(\sin y \sin z) \frac{\sin a}{\sin A} \cos \tfrac{1}{2}A$$

$$= \frac{n}{N} \Sigma \surd(\sin y \sin z) \cos \tfrac{1}{2}A;$$

therefore

$$\Sigma \surd(\sin y \sin z) \cos \tfrac{1}{2}A = -N = -2 \tan r \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$

(McClelland and Preston, chap. VI., Section 78).

The signs of the radicals have to be determined with reference to the three arcs of the in-circle separated by the three points of contact with the sides. For the arc which lies nearest the vertex A the true statement will be

$$-\surd(\sin y \sin z)\cos \tfrac{1}{2}A + \surd(\sin z \sin x)\cos \tfrac{1}{2}B + \surd(\sin x \sin y)\cos \tfrac{1}{2}C$$
$$= 2 \tan r \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

Note on Curvature.

By A. W. T.

If (x, fx) be a point on $y = f(x)$,

$$(X-x)^2 + (Y-fx)^2 = \lambda[(Y-fx) - f'x(X-x)]$$

represents a circle touching the curve at (x, fx); for it passes through the two consecutive points in which the tangent at (x, fx) cuts the point circle $(X-x)^2 + (Y-fx)^2 = 0$.

If it pass through a consecutive point $x+h$, $f(x+h)$,

$$h^2 + (hf'x + \ldots)^2 = \lambda\left(hf'x + \frac{h^2}{2!}f''x + \ldots - hf'x\right);$$

therefore $\lambda = 2\left\{[1 + (f'x)^2]/f'x\right\}$ in the limit. Hence the circle of curvature is determined. Similarly, by taking

$$(X-x)^2 - 2h(X-x)(Y-fx) + (Y-fx)^2 = \lambda[(Y-fx) - f'x(X-x)]$$

to be one of a series of rectangular hyperbolas having one-point contact with the curve, we can find the rectangular hyperbola of closest contact. Similarly with the other conics.

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9019. (Asparagus.)—In a given ellipse is inscribed a triangle ABC of maximum perimeter $(2s)$: prove that, if ρ_1, ρ_2, ρ_3 be the radii of curvature of the ellipse at A, B, C,

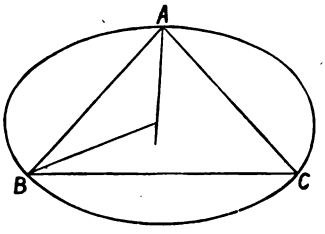
$$\rho_1 \cos^3 \tfrac{1}{2}A = \rho_2 \cos^3 \tfrac{1}{2}B = \rho_3 \cos^3 \tfrac{1}{2}C = s, \quad \rho_1\rho_2\rho_3 = 8r^3,$$

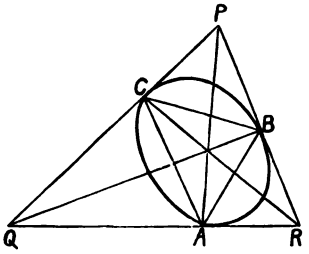
r being the radius of the circle inscribed in the triangle ABC.

[For s in the last line but one read $2s\Pi \sin \frac{1}{2}A$ (*vide* Solutions).—Ed.]

Solutions (I.) *by* W. C. Fletcher; (II.) *by* F. P. Andersen *and* Saradakanta Ganguli, M.A.

(I.) If the perimeter of ABC is a maximum, an ellipse whose foci are B, C and which passes through A will touch the given ellipse at A; *i.e.*, AB, AC are equally inclined to the normal at A; therefore the normals





at A, B, C are concurrent, and, if the tangents at A, B, C meet in P, Q, R, the latter are the escribed centres, or ABC is the pedal triangle of PQR.

Using trilinear co-ordinates and PQR as triangle of reference, the equation to the ellipse is

$$\alpha^2 \cos^2 P + \beta^2 \cos^2 Q + \gamma^2 \cos^2 R - 2\beta\gamma \cos Q \cos R - \ldots - \ldots = 0.$$

Transform this to Cartesian co-ordinates referred to AR, AP as axes by writing y for α,

$h \cos R - x \sin R - y \cos R$ for β, and $h \cos Q + x \sin Q - y \cos Q$ for γ,

and it reduces to the form $ux^2 + vxy + wy^2 - \lambda y = 0$,

where $u \equiv \sin^2 (Q + R)$, $\lambda = 4h \cos P \cos Q \cos R$.

But ρ, the radius of curvature at A, is $\lambda/2u = 2h \cos P \cos Q \cos R/\sin^2 P$, where $h = AP$; therefore

$$\rho \sin^3 P = 2h \sin P \cos P \cos Q \cos R = \tfrac{1}{4}D \sin 2P \sin 2Q \sin 2R,$$

where D is the diameter of the circle PQR. But the diameter of the circle ABC $= \frac{1}{2}D$ and $A = \pi - 2P$, ...; therefore

$$\rho \cos^3 \tfrac{1}{2}A = R \sin A \sin B \sin C;$$

therefore $\rho_1 \cos^3 \frac{1}{2}A = \rho_2 \cos^3 \frac{1}{2}B = \rho_3 \cos^3 \frac{1}{2}C = R \sin A \sin B \sin C$.

This, however, is not equal to s, as stated.

That the problem as stated is incorrect may be seen by taking the case of a line ellipse. In this case the triangle of maximum perimeter is a line, not a point, triangle; therefore s does not vanish. But, as ρ_1 may be the radius of curvature at the end of the line, ρ may vanish; so agreeing with the result $R \sin A \sin B \sin C$, but not agreeing with the stated result s or $4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

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A simple solution of the correct part of the problem is as follows:— The centre of the ellipse is the intersection of the lines through P, Q, R bisecting BC, CA, AB respectively; *i.e.*, the symmedian point whose co-ordinates are as sin P : sin Q : sin R. But, using the conventional lettering of geometrical conics, $\rho = CD^2/PF = a^2b^2/PF^3$; therefore

$$\rho_1 : \rho_2 : \rho_3 = 1/\sin^3 P : 1/\sin^3 Q : 1/\sin^3 R = 1/\cos^3 \tfrac{1}{2}A : 1/\cos^3 \tfrac{1}{2}B : 1/\cos^3 \tfrac{1}{2}C.$$

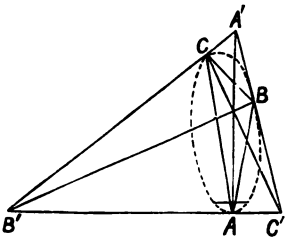
Lastly, $\rho_1\rho_2\rho_3 = R^3\, 8 \sin^3 \tfrac{1}{2}A\, 8 \sin^3 \tfrac{1}{2}B\, 8 \sin^3 \tfrac{1}{2}C = 8r^3$.

(II.) The condition that a triangle inscribed in an ellipse may be one of maximum perimeter is that the normals at the angular points bisect the angles of the triangle. Hence the ellipse is inscribed in the triangle A′B′C′, of which ABC is the pedal triangle.

Take A′B′C′ as the triangle of reference and use areal co-ordinates. Then the ellipse is

$$\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0,$$

where $l = b'c' \cos A'$, ...

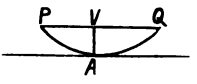


Near A write

$$y = n/(n+m) + \eta,\quad z = m/(n+m) - \eta - \kappa,$$
$$x = \kappa.$$

Consider the length of the chord PQ ($x = \kappa$), where κ is small. Neglecting squares of κ, we have

$$[(m+n)\,\eta + n\kappa]^2 - 2\kappa l\,[2mn/(m+n) + (m-n)\eta] = 0;$$



so that

$$(\eta_1 - \eta_2)^2 = 16lmn\kappa/(m+n)^3,$$

$$PQ^2 = a'^2(\eta_1 - \eta_2)^2 = a'^2\, 16lmn\kappa/(m+n)^3$$

$$(2s = a' \cos A' + \ldots = 4R' \sin A' \sin B' \sin C'),$$

$$\rho_1 = \operatorname*{Lt}_{\kappa=0} \frac{PQ^2}{8A\,V} = \frac{a'^2\, 16lmn}{(m+n)^3\, 16}\,\frac{a'}{\Delta'} \quad \left(4\Delta' = \frac{a'b'c'}{R'}\right)$$

$$= \frac{4R' \cos A' \cos B' \cos C' \sin A' \sin B' \sin C'}{\sin^3 A'}$$

$$(A = \pi - 2A',\ \ldots),$$

and therefore $\rho_1 \cos^3 \tfrac{1}{2}A = \ldots = \ldots = 2s \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$,

$$(\rho_1\rho_2\rho_3)^{\frac{1}{2}} = 2s \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C$$

$$= 2s\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{s^3(s-a)(s-b)(s-c)}} = 2\,\Delta/s = 2r,$$

$$\rho_1\rho_2\rho_3 = 8r^3.$$

15828. (*Communicated by* K. Deva Rao, B.A.)— A square board is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these squares be painted white, 4 black, 4 red, and

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4 green, without repeating the same colour in the same vertical or horizontal row?

[This familiar question is solved in a variety of ways. But the answer (576) may not be right.]

Solution by D. BIDDLE.

That 576 is the correct answer seems clear from the following considerations:—The colours of the top row can be arranged in 4!, or 24, different ways. Let the colours in any one of these 24 arrangements be designated by 1, 2, 3, 4 in order. Then the initial colours of the succeeding rows will be 2, 3, 4 variously arranged in 3!, or 6, different ways. For any one such arrangement there are four ways of filling in the remaining squares, as represented below:—

(i.)				(ii.)				(iii.)				(iv.)			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	3	4	1	2	4	1	3	2	1	4	3	2	1	4	3
3	4	1	2	3	1	4	2	3	4	1	2	3	4	2	1
4	1	2	3	4	3	2	1	4	3	2	1	4	3	1	2

As we have seen, the second, third, and fourth rows can be transposed in 6 different ways, each as a whole. Therefore the total number of ways in which the colours can be arranged, according to the conditions, is

$$24 \cdot 6 \cdot 4 = 576.$$

15792. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Sum to n terms the series $1 + 24 + 162 + 640 + 1875 + \ldots$.

Solutions (I.) *by* R. W. D. CHRISTIE; (II.) *by* JAMES BLAIKIE, M.A., KELA DEVA RAO, B.A., *and* F. W. REEVES, B.A.

(I.) The general term of the given series is $\frac{1}{2}[n^4(n+1)] = \frac{1}{2}(n^5 + n^4)$. Consequently the sum is $\frac{1}{2}(\Sigma n^5 + \Sigma n^4)$. Now, using the method I gave in *Knowledge*, 1889, and in Vol. XLIX., *Educational Times*, we have, starting from $\Sigma n^3 = \left\{\frac{1}{2}[n(n+1)]\right\}^2 = \frac{1}{4}(n^4 + 2n^3 + n^2)$,

the following rule:—Multiply by index of the highest power 4; then integrate and finally equate to unity when n is unity: thus $\int (n^4 + 2n^3 + n^2)$ becomes $\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \Sigma n^4$.

Again $\int (n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n)$ (after multiplying by index 5) becomes

$$\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \Sigma n^5.$$

Consequently the required sum is

$$\frac{1}{4}(\Sigma n^5 + \Sigma n^4) = \frac{1}{12}n^6 + \frac{7}{20}n^5 + \frac{1}{24}n^4 + \frac{1}{6}n^3 - \frac{1}{24}n^2 - \frac{1}{60}n.$$

(II.) $1 + 24 + 162 + \ldots = \frac{1}{2}(2 + 48 + 324 + \ldots)$

$= \frac{1}{2}(1 + 16 + 81 + \ldots) + \frac{1}{2}(1 + 32 + 243 + \ldots)$

$= \frac{1}{2}(\Sigma n^4 + \Sigma n^5)$,

which is known.

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15805. (A. H. Bell.)—Is 9462853 a prime number?

Solution by the Proposer.

If the number is not prime, to find its divisors might require testing with 440 prime trial divisors up to the limit $\sqrt{9462853} = 3076$, by actual division. The trial divisors can be reduced to a very small number by the method of Legendre.

The general complete quotient $[\sqrt{(kA)}+I]/D$ and the corresponding convergent p/q give the property $kAq^2 = p^2 \pm D$ or $= t^2 \pm au^2$, $D = au^2$. A divisor of A will also be one of $t^2 \pm au^2$.

From the square root of 9462853 or its multiples we have the formulæ t^2+3u^2, t^2+39u^2, t^2+21u^2, t^2-33u^2, t^2+23u^2, t^2-43u^2, ... The linear divisors for these formulæ are:

$t^2+3u^2 = 6x+1$ .....(1),

$t^2+39u^2 = 78x+$ *1*, 5, 11, *25*, 41, *43*, 47, *49*, *55*, 59, *61*, 71 .....(2),

$t^2+21u^2 = 84x+$ *1*, 5, 11, 17, 19, 23, *25*, *31*, *37*, 41, *55*, 71 .....(3),

$t^2-33u^2 = 132x+1, 25, 31, 37, 49 : 67, 91, 97, 103, 115$

[after combining with (1)] ...(4),

$t^2+23u^2 = 138x+1, 13, 25, 31, 49 : 55, 73, 85, 121, 127, 133$.....(5),

$t^2-43u^2 = 516x+1, 13, 25, 49, 97 : 109, 121, 133, 145, 169 : 181,$
$193, 229, 253, 289 : 325, 337, 361, 385, 397 : 445$...(6).

For the multiples of (1) by 13, in place of x, put successively

$$13x',\ 13x'+1,\ 13x'+2,\ \ldots,\ 13x'+(13-1).$$

Thus $t^2+3u^2 = 78x+1, 7, 13, 19, 25 : 31, 37, 43, 49, 55 : 61, 67, 73.$

Compared with (2) those in *italic* agree; the balance of the linear forms are rejected and the number of trial divisors are already reduced one-half. Treating (1) similarly with 14, and comparing with (3), there remain 5 linear forms. With (4) there remain 10 linear forms; using multiples of (5) by 3, and comparing with (1), there remain, as shown, 11; (6) is reduced from 63 linear terms to 21 as given. Comparing in the same way by multiples of each other (2), (3), (4), (5), (6), ..., there remain only six trial divisors, 607, 1039, 2467, 2473, (2539), and 2623; by actual division 2539 proves to be a factor, and $9462853 = 2539 \cdot 3727$, both prime factors.

Remark.—The six trial divisors are at the same time found in each of the linear forms of the six formulæ $t^2 \pm au^2$. The a of au^2, the denominator of the complete quotient, should be selected as small as possible, for on this depends the ease of development.

15823. (J. J. Barniville, B.A., I.C.S.)—Having $u_n = u_{n-1} + u_{n-2}$, prove that

$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{4.7} + \frac{2}{5.11} + \frac{3}{7.16} + \ldots = \frac{1}{2},$$

$$\frac{1}{1.2} - \frac{2}{2.4} + \frac{3}{4.5} - \frac{4}{5.7} + \frac{6}{7.11} - \ldots = \frac{1}{3},$$

$$\frac{2}{1.5} - \frac{3}{2.7} + \frac{5}{4.11} - \frac{7}{5.16} + \frac{10}{7.23} - \ldots = \frac{1}{4}.$$

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Solution by C. M. Ross.

Let $u_0, u_1, u_2, u_3, u_4, u_5, \ldots$ represent the series $1, 2, 4, 5, 7, 11, \ldots$; then

$$\frac{1}{1.4} = \frac{u_0}{u_0u_2} = \frac{1}{u_2},$$

$$\frac{1}{2.5} = \frac{u_0}{u_1u_2} = \frac{u_2-2u_1}{u_1u_2} = \frac{1}{u_1}-\frac{2}{u_2},$$

$$\frac{1}{4.7} = \frac{u_0}{u_2u_4} = \frac{2u_2-u_4}{u_2u_4} = \frac{2}{u_4}-\frac{1}{u_2},$$

$$\frac{2}{5.11} = \frac{u_1}{u_3u_5} = \frac{2u_3-4u_2}{u_3u_5} = \frac{2}{u_3}-\frac{4}{u_5},$$

$$\frac{3}{7.16} = \frac{u_0+u_1}{u_4u_6} = \frac{5u_4-u_6}{u_4u_6} = \frac{5}{u_6}-\frac{2}{u_4},$$

and so on; therefore $S = 1/u_1 = \frac{1}{2}$, all the other terms vanishing. Similarly in the second series

$$\frac{1}{1.2} = \frac{1}{6}+\frac{1}{3}, \quad \frac{2}{2.4} = \frac{1}{6}+\frac{1}{12}, \quad \frac{3}{4.5} = \frac{1}{12}+\frac{1}{15},$$

$$\frac{4}{5.7} = \frac{1}{15}+\frac{1}{21}, \quad \frac{6}{7.11} = \frac{1}{21}+\frac{1}{33}, \quad \ldots;$$

therefore $S = \frac{1}{2}$; and finally, for the third series,

$$\frac{2}{1.5}-\frac{3}{2.7}+\frac{5}{4.11}\ldots = \frac{1}{2}-\frac{2}{5}-\frac{5}{7}+\frac{1}{2}+\frac{4}{11}-\frac{1}{4}-\frac{2}{5}+\frac{5}{16}+\frac{5}{7}-\frac{15}{23}-\ldots$$

$$= \frac{1}{4}.$$

15826. (J. A. C., M.A.) — Solve, in integers, $x^2+y^2 = 5z^2$; and generally solve $x^2+y^2 = Az^2$, where A is not a square number.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.; (II.) *by* Alexander Holm, M.A.

(I.) Take the general case $x^2+y^2 = Az^2$ first. The possibility of this requires that $\tau^2-Av^2 = -1$ should be possible, and the actual solution (x, y, z) requires that τ, v should be known. Take y', z' so that $y'^2-Az'^2 = x^2$. The general solutions of this Diophantine are known to be of two kinds:

$$y' = \tfrac{1}{2}(t^2+Au^2), \quad z' = tu, \quad x = \tfrac{1}{2}(t^2-Au^2) \ldots\ldots\ldots\ldots \text{(i.)};$$

$$y' = t^2+Au^2, \quad z' = 2tu, \quad x = t^2-Au^2 \ldots\ldots\ldots\ldots\ldots \text{(ii.)}.$$

Here in (i.) t, u must be both odd or both even integers; whilst in (ii.) t, u many be any integers. Also $(y'^2-Az'^2)(\tau^2-Av^2) = -x^2$, *i.e.*,

$$(\tau y' \mp Av z')^2 - A(\tau z' \mp v y')^2 = -x^2 = y^2-Az^2,$$

whence $y = \tau y' \mp Av z'$, $z = \tau z' \mp v y'$ [y', z' as in (i.) or (ii.) above]. These are the general solutions [using both +, or both −, signs in above].

Example.—Take $A = 5$; then $2^2-5.1^2 = -1$, $\tau = 2$, $v = 1$. The

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general solution is $y = 2y' \mp 5z'$, $z = y' \mp 2z'$, where y', z', x are given by either set of $y' = \frac{1}{2}(t^2 + 5u^2)$, $z' = tu$, $x = \frac{1}{2}(t^2 - 5u^2)$ ..... (i.);
$y' = t^2 + 5u^2$, $z' = 2tu$, $x = t^2 - 5u^2$ ..... (ii.),

t, u being integers as above stated.

(II.) Assuming x and y to be prime to one another, the solution in integers of $x^2 + y^2 = Az^2$ is possible only when A is a square or the sum of two squares. For, A being a factor of $x^2 + y^2$, the sum of two prime squares must be a square or the sum of two squares (Fermat). Taking $A = a^2 + b^2$, we have $x^2 + y^2 = (a^2 + b^2)z^2$; therefore

$$x^2 - b^2z^2 = a^2z^2 - y^2 \quad \text{or} \quad (x + bz)(x - bz) = (az + y)(az - y);$$

therefore

$$(x + bz)/(az + y) = (az - y)/(x - bz) = \text{ratio of two integers} = m/n \text{ (say)};$$

therefore

$$nx - my + (-am + bn)z = 0 \quad \text{and} \quad mx + ny - (bm + an)z = 0;$$

therefore

$$x/[a(2mn) + b(m^2 - n^2)] = y/[b(2mn) - a(m^2 - n^2)] = z/(m^2 + n^2) = l \text{ (say)}.$$

This might be deduced from the solution in integers of $x'^2 + y'^2 = z^2$. For then we should have

$$(a^2 + b^2)(x'^2 + y'^2) = (a^2 + b^2)z^2 \quad \text{or} \quad (ax' + by')^2 + (bx' - ay')^2 = (a^2 + b^2)z^2.$$

In the particular case $a = 2$, $b = 1$.

13986 & 15797. (13986.) (Professor Nanson.) — If $t_1, t_2, \ldots, t_m$ are the m tangents and $n_1, n_2, \ldots, n_{m+1}$ are the $m + 1$ normals from any point to the curve $y^m = ax^{m-1}$, prove that $m^m n_1 n_2 \ldots n_{m+1} = at_1 t_2 \ldots t_m$.

(15797.) (*Communicated by* M. V. A. Sastry, B.A.)—If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to a semi-cubical parabola $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$. [An elegant solution wanted.]

Solution by Professor Nanson.

The points of contact of the tangents and the feet of the normals from $\xi\eta$ to $x^{m+n} = a^m y^n$ .....(1)

lie on the conics $(m + n)y(x - \xi) - nx(y - \eta) = 0$ .....(2),

$(m + n)y(y - \eta) + nx(x - \xi) = 0$ .....(3),

respectively. To find the lengths and directions of the tangents and normals, let $\rho = z - \zeta$, where $z = x + \iota y$, $\zeta = \xi + \iota\eta$. Thus (1), (2), (3) become

$$\iota^n x^{m+n} - a^m(\rho + \zeta - x)^n = 0 \text{ .....(4)},$$

$$mx^2 - [m(\rho + \xi) + (m + n)\zeta]x + (m + n)(\rho + \zeta)\xi = 0 \text{.....(5)},$$

$$mx^2 - [2(m + n)\rho + m\xi + (m + n)\zeta]x + (m + n)(\rho + \zeta)(\rho + \xi) = 0 \text{ ...(6)}.$$

By substituting for x in (4) the two roots of (5) or (6) and multiplying the results, we get an equation of the form $P_s\rho^s + \ldots + P_0 = 0$ for the determination of the ρ of a tangent or normal. Now, since (5), (6) are

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identical when $\rho = 0$, it follows that P_0 is the same in the two cases. To find P_s we take ρ infinite. The values of x/ρ given by (5) are then 1 and 0. Hence for the eliminant of (4), (5) we have

$$P_s\rho^s \equiv \iota^n\rho^{m+n}(-a^m\rho^n);$$

and therefore $s = m+2n$, $P_s = -a^m\iota^n$. Again, when ρ is infinite the two values of x/ρ given by (6) are finite both ways and their product is $1+n/m$. Hence for the eliminant of (4), (6), we have

$$P_s\rho^s = \iota^{2n}\rho^{2(m+n)}(1+n/m)^{n+m};$$

and therefore $s = 2m+2n$, $P_s = \iota^{2n}(1+n/m)^{m+n}$. Hence, if T_ρ, N_ρ are the products of the $m+2n$, $2m+2n$ values of ρ given by the two eliminants,

$$T_\rho : N_\rho = (1+n/m)^{m+n} : (-1)^{m-1}\iota^n a^m.$$

Now, amongst the values of ρ given by each eliminant are n values drawn to the origin because the conics (2), (3) each cut (1) in n points at the origin. Hence, if T, N are the products of the $m+n$ tangents and the $m+2n$ normals drawn to ordinary points of (1), then

$$(1+n/m)^{m+n}N = a^mT.$$

Also, if τ, ν are the sums of the angles made with the axis of y by the tangents and normals respectively, then $\nu-\tau = (4k+m+n-2)\frac{1}{2}\pi$, where k is an integer. Taking $m = 1$, we get 13986; taking also $n = 2$, we get 15797; whilst $n = 1$ gives well known results for the common parabola.

15832. (Professor Neuberg.) — Le centre A d'une sphère de rayon constant R parcourt une droite donnée d. Trouver (1) l'enveloppe du plan polaire π d'un point fixe P par rapport à la sphère mobile A, (2) la surface engendrée par la circonférence suivant laquelle le plan π coupe la sphère.

Solution by S. C. Bagchi, B.A., *and* C. M. Ross.

The centre of the sphere has a single degree of freedom. Its variability in position will, therefore, be expressed by the presence of a single parameter in its equation. Take the origin O coincident with one point of $d \equiv OZ$. Now the polar plane will contain a variable parameter in the second degree. Therefore the envelope itself will be a surface of the second degree.

(1) The sphere is given by

$$x^2+y^2+z^2-2az-R^2+a^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

The polar plane with regard to P (ξ, η, ζ) is

$$x\xi+y\eta+z\zeta-a(z+\zeta)+(a^2-R^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

The condition that (a) may have two equal roots in a is given by

$$(z+\zeta)^2 = 4(x\xi+y\eta+z\zeta-R^2),$$

which is the envelope required, a parabolic cylinder.

(2) The surface engendered by the circle is a tubular one which is obtained by eliminating a between (a) and (b). It has for its equation

$$(x^2+y^2+z^2-x\xi-y\eta-z\zeta)^2 = (\zeta-z)[2z(x\xi+y\eta+z\zeta-R^2)-(z+\zeta)(x^2+y^2+z^2-R^2)].$$

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15780. (D. Biddle.)—PQ is a chord of a given circle; LOM is an intercept of the ellipse which, having P, Q as its foci, passes through O, the centre of the circle, L, M being on the circumference. (1) Find the position of PQ when the equal triangles PQL, PQM are maxima; and their area, as such. (2) Consider similarly the cases in which the ellipse cuts the circumference in S, T as well, on the opposite side of P, Q, and in which (for the limited range) PQS, PQT exceed in area PQL, PQM, the corresponding triangles previously considered.

Solution by L. Isserlis, B.A.

(1) Draw ONC at right angles to PQ, LM.

a = radius of circle,

b = OC.

Then $PC^2 = a^2 - b^2$,

$LN^2/a^2 + CN^2/b^2 = 1$,

and $LN^2/a^2 + ON^2/a^2 = 1$;

therefore

$CN = (b/a)\,ON$

$= (b/a)(b - CN)$

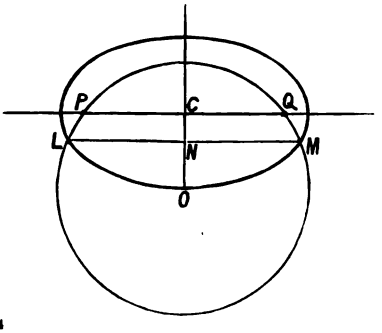
$= b^2/(a+b)$;

therefore

(area of $\triangle PQL)^2$

$= PC^2 \cdot CN^2 = [(a-b)/(a+b)]\,b^4$

= maximum.



Therefore, differentiating logarithmically, $1/(a-b) + 1/(a+b) = 4/b$; therefore $2b^2 + ab - 2a^2 = 0$; therefore $b = \frac{1}{4}(\sqrt{17}-1)\,a$ when the area is a maximum, and area $= \sqrt{[(a-b)/(a+b)]}\,b^2$.

(2) $CN = (b/a)\,ON$

$= (b/a)(b + CN)$

$= b^2/(a-b)$;

therefore

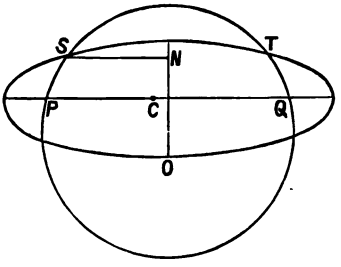
$PC^2 \cdot CN^2 = [(a+b)/(a-b)]\,b^4$,

giving similarly

$2b^2 - ab - 2a^2 = 0$

or $b = \frac{1}{4}(\sqrt{17}+1)\,a$.

But this makes $b > a$, which is impossible; for in the present case $b \not> \frac{1}{2}a$. It indicates, however, that the area increases with b to the limit that is possible. Therefore $b = \frac{1}{2}a$, and area $= \sqrt{[(a+b)/(a-b)]}\,b^2 = \frac{1}{4}\sqrt{3}\,a^2$.



15803. (Professor Neuberg.) — ABCD étant un rectangle donné, trouver sur le côté AD un point X tel que les volumes engendrés par le triangle ABX et le trapèze BCDX tournant autour de la droite BX aient un rapport donné k.

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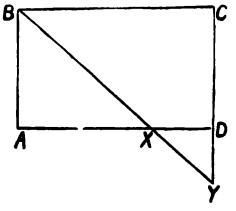
Solution by A. M. NESBITT, M.A.

The volumes of the solids generated respectively by the revolutions of the triangles ABX, BCY, XDY about BY are proportional to the cubes of AX, BC, XD.



Hence, if $AX = x$, $AD = a$,

we have at once

$$x^3 = k[a^3-(a-x)^3],$$

yielding the quadratic

$$x^2(1-k)+3kax = 3a^2k.$$

The positive root gives the value sought for x.

A short way of finding the Recurring Decimal Fraction equal to a given Vulgar Fraction.

By W. D. Ross.

Let m/n be a fraction such that n is not a multiple of 2 or of 5. Find, by inspection, x and y such that $nx = 10y-1$. Then

$$m/n = mx/(10y)+mx/(100y^2)+mx/(1000y^3)+\ldots \text{ ad inf.}$$

This gives a practical rule. The first figure of the decimal will be the highest integer less than mx/y. Multiply the remainder by 10 and add the first figure, and divide the sum by y. The highest integer less than this quotient will be the second figure of the decimal fraction, and so on. Divide always by y, *e.g.*, $\frac{3}{19}$. Here $m = 3$, $n = 19$, $x = 1$, $y = 2$. Hence $\frac{3}{19} = \cdot\dot{1}57894736842105263\dot{1}$. Each figure is got by dividing by 2, not as in the ordinary method by dividing by 19.

We can, with equal ease, start from the right-hand figure. This figure is mx, or, if mx is greater than 10, it will be the unit figure of mx. Divide the remaining part (if any) of mx by 10, add in the right-hand figure, and multiply the sum by 10. The product, or the unit figure of the product, will be the second figure of the decimal (counting from the right-hand), and so on. Multiply always by y, *e.g.*, $\frac{3}{7}$. Here $m = 3$, $n = 7$, $x = 7$, $y = 5$. Hence $\frac{3}{7} = \cdot\dot{4}28571\dot{4}$.

These processes are much simpler in practice than they look when described abstractly. When the numbers are large the method saves much labour, especially when the denominator ends in 1 or in 9. Such a fraction as $\frac{13}{89}$ can be written with ease to as many places as are wanted: $x = 1$, $y = 9$—

$$\frac{13}{89} = \cdot 1460674157\ldots$$

or, from the end, $\ldots 2696629213483\dot{1}$.

10730. (Professor DÉPREZ.)—Trouver l'enveloppe des axes des coniques touchant les côtés AB, AC d'un triangle, et ayant pour asymptote le troisième côté BC.

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Solution by J. A. H. JOHNSTON, M.A.

Let the conic touch the lines $y = mx$ and $y = m'x$, and let the given asymptote be $x = a$. Its equation is $(x-a)(px+qy+1)+r = 0$. If $y = mx$ and $y = m'x$ be tangents, then

$$[1-a(p+qm)]^2 = 4(r-a)(p+qm), \quad [1-a(p+qm')]^2 = 4(r-a)(p+qm').$$

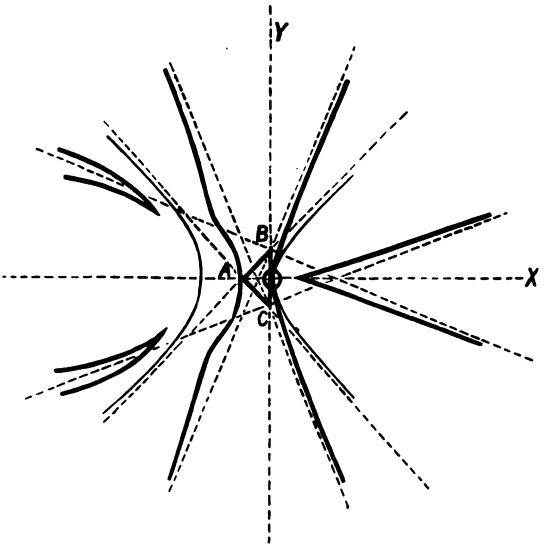
Eliminate r, and $a^2(p+mq)(p+m'q)-1 = 0$ .....(i.).

(i.) shows that the other asymptote $px+qy+1 = 0$ envelops the conic

$$mm'x^2-(m+m')xy+y^2+\tfrac{1}{4}a^2(m-m')^2 = 0 \text{ .....(ii.).}$$

This conic touches $x = a$, where $y = \frac{1}{2}(m+m')a$. Transfer to parallel axes through this point, and this conic becomes

$$mm'x^2-(m+m')xy+y^2-\tfrac{1}{2}(m-m')^2ax = 0 \text{ .....(iii.).}$$



We have simply to find the envelope of the bisectors of the angle between $x = 0$ and a variable tangent to (iii.). If we choose κ as parameter, where $x = 0$, $y = \kappa$ is the intersection of this tangent with $x = 0$, it readily appears that $x[4\kappa^2+4(m+m')a\kappa+(m-m')^2a^2] = 8a\kappa(y-\kappa)$ is the equation of this variable tangent. Then $(y-\kappa)/x = \cot\phi$ is a bisector, where $\tan 2\phi = 8a\kappa/[4\kappa^2+4(m+m')a\kappa+(m-m')^2a^2]$.

If we eliminate ϕ, a typical pair of bisectors is given by

$$\kappa^3(4x+4a)+\kappa^2[4(m+m')ax-4xy-8ay]$$
$$+\kappa[4a(y^2-x^2)+(m-m')^2a^2x-4(m+m')axy]-(m-m')^2a^2xy = 0\text{...(iv.).}$$

Divide (iv.) by $4a^4$. Let $\kappa/a = \lambda$, $x/a = X$, $y/a = Y$, $(m+m') = S$, and $\frac{1}{4}(m-m')^2 = p$. Then a pair of axes is given by

$$\lambda^3(X+1)+\lambda^2(SX-2Y-XY)+\lambda(Y^2-X^2+pX-SXY)-pXY = 0\text{...(v.).}$$

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The discriminant of this cubic in λ equated to zero then gives the desired envelope, which in this general case is

$$(SX-2Y-XY)^2(Y^2-X^2+pX-SXY)^2$$
$$-18pXY(X+1)(SX-2Y-XY)(Y^2-X^2+pX-SXY-27p^2X^2Y^2(X+1)^2$$
$$-4(X+1)(Y^2-X^2+pX-SXY)^3+4pXY(SX-2Y-XY)^3=0 \quad \text{.....(vi.).}$$

On expansion and examination, this envelope is a sextic touching the x-axis at the origin, having a parabolic asymptote and four linear asymptotes which are the external and internal bisectors of the angles between the original pair $y=mx$ and $y=m'x$ and $x=a$ (former axes), and having also three cusps. It, moreover, touches $y=0$.

Its chief characteristics may be exemplified, with diminished labour, by the special case when the two given tangents and asymptote of the original conic form an isosceles right-angled triangle. Then $S=m+m'=0$ for $m=1$, and $m'=-1$, $p=1$; the conic enveloped by the second asymptote is $x^2-y^2+2ax=0$, and the envelope of the axes (vi.) reduces at once to

$$y^2(y^4+x^4-6x^2y^2)-4x^3(7y^2-x^2)+(6y^4-8x^4-54x^2y^2)$$
$$-28xy^2+y^2+8x^2-4x=0 \quad \text{.....(vii.),}$$

where y and x are used for Y, X, and a may be unity. If $y=0$, $x=0$, $x=-1$, and $x=1$ repeated thrice, there is a cusp at $x=1$, and the shape of the curve appears as in the diagram.

The general shape of the curve differs from this only in lack of symmetry and in the position of the cusp ($x=1$). It may be noted also that the cusps in the parabolic branch appear when one axis-bisector is passing through a point of inflexion on the neighbouring infinite branch, the other axis-bisector then being stationary at the cusp. The whole process of generation of the envelope may be simply studied from the diagram, where the interior infinite branches are seen to be the envelope of the internal bisectors and the cuspidal branches the envelope of the external bisectors of the angle between $x=0$ and a variable tangent to the asymptote envelope hyperbola.

15885. (The late R. Tucker, M.A.)—Given A, a, b in a triangle (the "ambiguous" case), if ω, ω' are the "B" angles of the triangle, prove

$$\cot\omega+\cot\omega'=2b^2\cot A/(cc').$$

Solutions (I.) *by the* Proposer; (II.) *by* M. V. A. Sastry, B.A., Kela Deva Rao, B.A., *and* K. S. Patrachan.

(I.) $\cot\omega+\cot\omega'=\cot A+\cot B-\cot(A+B)+\cot A-\cot B$
$$+\cot(B-A)$$
$$=2\cot A+2\sin A\cos A/(\sin^2 B-\sin^2 A)$$
$$=2\cot A\{\sin^2 B/[\sin C\sin(B-A)]\}$$
$$=2\cot A\, b^2/(cc').$$

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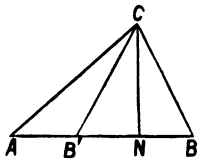
(II.) Let ABC and AB'C be the two triangles in the ambiguous case, and let CN be perpendicular on AB.

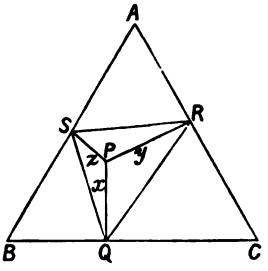


$$\cot\omega + \cot\omega' = \cot A + \cot B + \cot C + \cot A + \cot B' + \cot C'$$
$$= 2\cot A + \cot C + \cot C' \quad (\text{since } \cot B' = -\cot B)$$
$$= 2\cot A + \frac{\sin(C+C')}{\sin C \sin C'} = 2\cot A + \frac{\sin 2ACN}{cc' \sin^2 A/a^2}$$
$$= 2\cot A + \frac{a^2 \sin 2A}{cc' \sin^2 A} = 2\cot A + 2a^2 \cot A/(cc')$$
$$= \frac{2\cot A}{cc'}(cc' + a^2) = 2b^2 \cot A/(cc').$$

15811. (R. Chartres.)—From a variable point within a triangle perpendiculars are drawn to the sides. Find (1) the maximum value, (2) the mean value of the product of the perpendiculars, and (3) the mean value of the area of the triangle formed by joining the feet of the perpendiculars. [Elementary proof.]

Solution by the Proposer.

Let the triangles BPC, CPA, APB be denoted by α, β, γ.



(1) Since $\Delta = \alpha + \beta + \gamma$,
therefore the maximum value of $(\alpha\beta\gamma)$ is when $\alpha = \beta = \gamma$ or the maximum value of xyz is at the centroid.

(2) $\Delta^3 = (\alpha+\beta+\gamma)^3$, having ten terms; therefore

$$M(\alpha\beta\gamma) = \tfrac{1}{60}\Delta^3;$$

therefore

$$M(xyz) = \tfrac{1}{60}\Delta^3/\tfrac{1}{8}abc = 2\Delta^3/15abc.$$

(3) $\Delta^2 = (\alpha+\beta+\gamma)^2$; therefore

$$M(\beta\gamma) = \tfrac{1}{12}\Delta^2;$$

therefore $\quad M(\text{RPS}) = \tfrac{1}{2}(\sin A)\, M(yz) = \tfrac{1}{12}(\Delta \sin^2 A),$

or $\quad M(\text{QRS}) = \tfrac{1}{12}\Delta \cdot \Sigma(\sin^2 A).$

15835. (Professor Cochez.)—Trouver le lieu des foyers d'une hyperbole dont on connait un sommet et une asymptote.

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Solution by C. M. Ross.

Take the given asymptote and a perpendicular through the vertex as axes; then, if the equation of the curve is

$$ax^2+by^2+2hxy+2gx+2fy+c=0,$$

the points where it is met by the asymptote are given by $ax^2+2gx+c=0$. This equation has both roots infinite if $a=0$ and $g=0$; thus the equation of the curve reduces to $by^2+2hxy+2fy+c=0$. Again, if (0, K) is the vertex, $bK^2+2Kf+c=0$ ..... (1).

Now the equations

$$[(hy)^2-(by+hx+f)^2]/-b = hy(by+hx+f)]/h = f(x, y)$$

give the foci, and on reduction become

$$h^2(x^2-y^2)+f^2+2hfx-bc=0, \quad hxy+fy+c=0 \ldots\ldots(2, 3).$$

Also $(-f/h, 0)$ is the centre of curve; therefore, since (x, y) is a focus,

$$y/(x+f/h) = K/(f/h) \ldots\ldots (4).$$

From (1) and (3), $bK^2+2fK-hxy-fy=0$ ..... (5).

(2) reduces to $h^2(x^2-y^2)+f^2+2hfx+b(hxy+fy)=0$ .....(6).

Again, from (5), $b=(hxy+fy-2fK)/K^2$. Substituting in (6),

$$K^2[h^2(y^2-x^2)-f^2-2hfx] = y(hx+f)[y(hx+f)-2fK].$$

Dividing by h^2,

$$K^2(y^2-x^2)-x^2y^2+2f(Kxy-xy^2-K^2x)/h-f^2(K^2+y^2-2Ky)/h^2=0.$$

But by (4) it reduces to

$$K^2(x^2+y^2)+K^2[x^2(1-y)-y^3]+(K+y)x^2y^2=0,$$

the required locus of the foci.

15768. (S. C. Bagchi, B.A.)– The rate of change of the Massieu-characteristic of a homogeneous mixture due to a small amount of one of its constituents being allowed to diffuse in it is invariantive with regard to the proportional increase in the molecular weight of each constituent. (Massieu-characteristic is called by Duhem the "thermodynamic potential at constant pressure.")

Solution by the Proposer.

In the case of a homogeneous mixture, it is well known (*vide* Ariès, *Statique Chimique*) that the Massieu-characteristic may be represented by a homogeneous function of $x_1, x_2, \ldots$, where these are the proportions of the molecular weights of several components, and p the pressure, and T the absolute temperature. The given theorem is tantamount to showing that a homogeneous function is invariantive as regards the partial derivative, which is easily shown. Let $F(x_1, x_2, \ldots)$ be such a function. Then $[\partial F(kx_1, kx_2, \ldots)]/[\partial(kx_1)] = k[\partial F/\partial(kx_1)] = \partial F/\partial x_1$, and so for any other of the arguments. Hence the result follows at once.

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15830. (R. CHARTRES.)—A point P is taken within a square and joined to A, the extremity of the diagonal AC. Find the mean value (1) of the angle PAC, and (2) of its square.

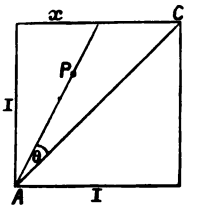
Solution by the PROPOSER.

$$M(\theta) = \tfrac{1}{4}\pi - \int_0^1 \tan^{-1} x\,dx \Big/ \int_0^1 dx$$

$$= \tfrac{1}{4}\pi - \tfrac{1}{4}\pi + \tfrac{1}{2}\log 2 = \tfrac{1}{2}\log 2.$$

$$M(\theta^2) = \tfrac{1}{16}\pi^2 - \tfrac{1}{2}\pi\,(\tfrac{1}{4}\pi - \tfrac{1}{2}\log 2) + \int_0^1 (\tan^{-1} x)^2\,dx$$

$$= -2\int_0^{\frac{1}{4}\pi} \log\cos\phi\,d\phi.$$



15837. (Professor SANJÂNA, M.A.)—Points P, Q, R are taken in the sides BC, CA, AB of a triangle such that the areas AQR, BRP, CPQ are each equal to a given area T. Shew how to construct for the points, and prove that the distances of any two from the side not containing them are together equal to the corresponding altitude of the triangle.

Solutions (I.) *by* H. M. TAYLOR, F.R.S.; (II.) *by* HENRY RIDDELL, M.E.

(I.) Let the ratios of BP to BC, CQ to CA, AR to AB be represented by x, y, z respectively. Then the ratios of the three triangles PCQ, QAR, RBP to the triangle ABC will be equal to $(1-x)y$, $(1-y)z$, $(1-z)x$ respectively. Because the triangles are equal, we have

$$(1-x)y = (1-y)z = (1-z)x = k.$$

If we eliminate y and z from the above equations, we obtain

$$(1-x-k)(x-k) = (x-x^2)k \quad \text{or} \quad (k-1)x^2-(k-1)x+k^2-k = 0.$$

Therefore, if $k \neq 1$, we have $x^2-x+k = 0$. Therefore

$$x = \tfrac{1}{2}[1 \pm \sqrt{(1-4k)}].$$

Here x is real for every positive value of k not greater than $\frac{1}{4}$, and for every negative value.

It will be observed that for any such value of k we must have the equalities $x = y = z$. For any such value of x, y, z the algebraical sum of the distances of P and Q from AB is equal to the distance of C from AB; in this case the triangle PQR has the same centre of gravity as the triangle ABC. If, however, $k = 1$, then x may be chosen arbitrarily, and then y and z are determined by the equations

$$(1-x)y = 1, \quad (1-z)x = 1.$$

It will be observed that so far the signs of the three triangles AQR, BRP, CPQ have been assumed to be the same. There is, however, another case to be considered, *i.e.*, that in which one of the triangles, say

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AQR, has a sign different from that of the other two BRP, CPQ. In this case we have to solve the equations

$$-(1-x)y = (1-y)z = (1-z)x = k.$$

If we eliminate y and z from these equations, we obtain

$$-(1-z-k)(z-k) = (y-z^2)k \quad \text{or} \quad (k+1)z^2-(k+1)z-k^2+k = 0,$$

whence $$2(k+1)z = k+1 \pm s,$$

where $$s^2 = 4k^3+k^2-2k+1 = (k+1)(4k^2-3k+1).$$

We see therefore that z has a real value for any positive value of k, and also for any negative value between 0 and -1.

When one of the two possible values of z for a given value of k has been chosen the corresponding values of x and y are determined by the equations $(1-y)z = k$ and $(1-z)x = k$.

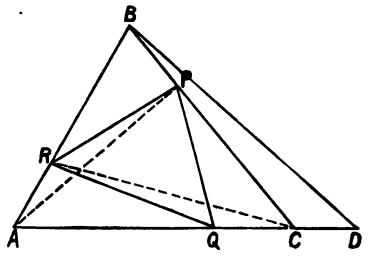
(II.) Produce the side AC, making area BCD = T. Divide AC in Q, so that

AQ . QC = AC . CD.

Take points R and P, so that AR/RB = BP/PC = CQ/QA. These are the points required. For, if CQ/QA = (say) m/n,

area BCD = CD/AC . Δ

(where Δ = area ABC)



$$= [(CD \cdot AC)/AC^2]\,\Delta = [(CQ \cdot QA)/AC^2]\,\Delta = [mn/(m+n)^2]\,\Delta.$$

And the three triangles AQR, BRP, and CPQ have areas similarly equal $[mn/(m+n)^2]\,\Delta$. This is seen at once by comparing the rectangles under corresponding pairs of sides about common angles.

For the second part of the problem, join AP and RC. Then area ABP = area ACR, since each = $[m/(m+n)]\,\Delta$; therefore

$$\text{area ACR} + \text{area ACP} = \Delta \quad \text{or} \quad AC \cdot p_1 + AC \cdot p_2 = AC \cdot p_0,$$

where p_1 and p_2 are altitudes of ACR, ACP, and p_0 is altitude of ABC. Therefore $p_1 + p_2 = p_0$.

There are four positions of Q for which AQ . QC = AC . CD, two internal to AC and two external. When $T > \frac{1}{4}\Delta$ the internal positions are no longer real. In the case of the external points the areas cut off are wholly outside ABC.

15799. (D. Biddle.)—From a point A on the circumference of a given circle a straight line of diameter's length is drawn in any direction in the same plane. (1) Find, on the circumference, another point P such that a straight line drawn from it through the centre shall meet the given line in X, making AX = AP. (2) Find the locus of X.

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Solutions (I.) *by* H. J. RAYMOND; (II.) *by* JAMES BLAIKIE, M.A.; (III.) *by* ALEX. MATHESON, M.B., Ch.B.

(I.) (1) Let O be the centre, OA the radius of the given circle, and B_1AB the given straight line. On either side of OA draw the straight line AP to meet the circumference in the required point P, such that the angle OAP shall be equal to one-third of that between B_1AB and OA. Join AP, PO and produce PO to meet B_1AB in X. The angles OAP, APO are evidently equal to each other. By construction

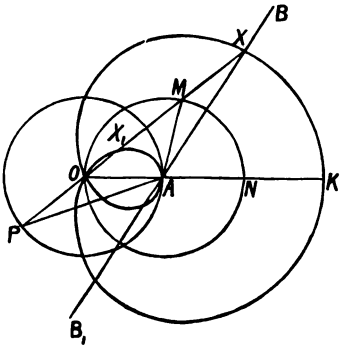
$\angle B_1AP = 2\angle OAP$ or $2\angle APO$;

and therefore

$$\angle AXO = \angle APO$$

(Euclid I. 32). Hence the triangle APX is isosceles, and

$$AP = AX.$$



(2) On the line PX take $MX = OP = OA$. Then $AM = AO =$ constant b. The locus of M is a circle, centre A, and diameter ON = constant a. Taking O as pole, $OX = r$ as radius vector, with initial position OK, $\angle XOA = \theta$ and $MX_1 = MX = AO = b$, on either side of M along OX; the locus of X is the limacon given by $r = a\cos\theta + b$.

The constant a being greater than that of b, the limacon in this case becomes the inverse of a hyperbola with respect to a focus at the pole O, as the centre of inversion, the eccentricity of the conic being a/b, and the asymptotes inclined at 60° to the initial line OK_1 along which lies the major axis.

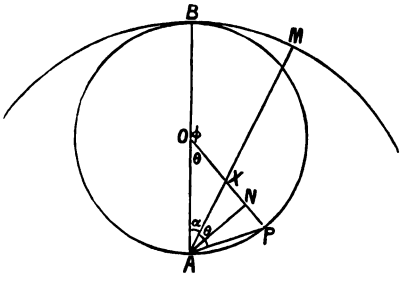
If A and O be the centres of the fixed and rolling circles respectively, whose radii are each equal to $\frac{1}{3}OA$, A being the initial position of the tracing point X, then the locus of X, a limacon, is a special case of the epitrochoid. Since $a = 2b$ in this limacon, the curve may be used to trisect any given angle.

(II.) (1) Take any point P on the circumference. Join OP and draw AN perpendicular to OP; take NX on NO = NP. Then X is the point on the line AX. Also

$$\angle AOX = \angle PAX,$$

since the triangles OAP, AXP are both isosceles triangles having an angle at the base common. Therefore

$$\angle OAX = \tfrac{1}{2}\pi - \tfrac{1}{2}\theta = \alpha;$$



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therefore $\theta = \frac{1}{3}(\pi - 2\alpha)$. Thus, if the straight line AM be given, we know α and therefore θ and can then find X and P.

(2) To find the locus of X.—Let OX = r, OA = a; then

$$ON = \tfrac{1}{2}(a + r) = a\cos\theta;$$

therefore $\quad a + r = 2a\cos\theta$ and $r = a(2\cos\theta - 1)$.

This is the equation to the trisectrix, a special case of the limaçon of Pascal, and the inverse of the hyperbola whose eccentricity is 2.

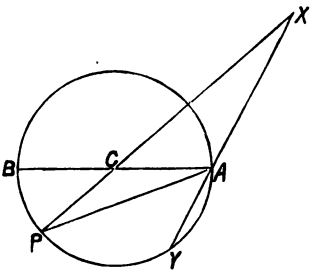
(III.) (1) Let AX cut the circle again in Y. Join AC and produce to meet the circle in B. If AP = AX (in figure), then

$$\angle PAY = 2\angle APX = 2\angle PAB;$$

therefore $\quad \angle PAB = \frac{1}{3}\angle YAB$.

Therefore the construction required is as follows:—Trisect the arc BY and let arc BP = $\frac{1}{3}$ arc BY. Join PC and produce to meet the given line through A in X. Then

AP = AX.



(2) To find the locus of X. Take A as pole, AB as the fixed direction, and AX as the radius vector. Let AB = a and

$$\angle BAX = \theta.$$

Then $AX = AP = a\cos\frac{1}{3}(\pi - \theta)$;

that is, the locus of X is given by the equation $r = a\cos\frac{1}{3}(\pi - \theta)$.

It is interesting to note that, if AX remains fixed in direction and P moves round the circle while AX always equals AP, then the line PX envelops a tricuspid hypocycloid.

Editorial Note.

In *The Educational Times* for October, 1905, and on p. 73 of the present volume of the *Reprint*, we gave an interesting theorem connected with recurring decimals and from the pen of Mr. W. D. Ross. The theorem, we find, must be added to the list of those which from time to time have been discovered independently by more than one mathematician, for Mr. R. W. D. Christie writes that the same theorem appeared as one of a series which he contributed to *Knowledge*, "Some Properties of Numbers," Feb. 1, 1890. *Cf. Reprint*, 1st Series, Vol. LXIX., p. 104, &c.

15787. (W. J. Dobbs, M.A.)—Show how to resolve a given force into three forces of given magnitudes acting respectively through three given points situated in one plane with the line of action of the given force.

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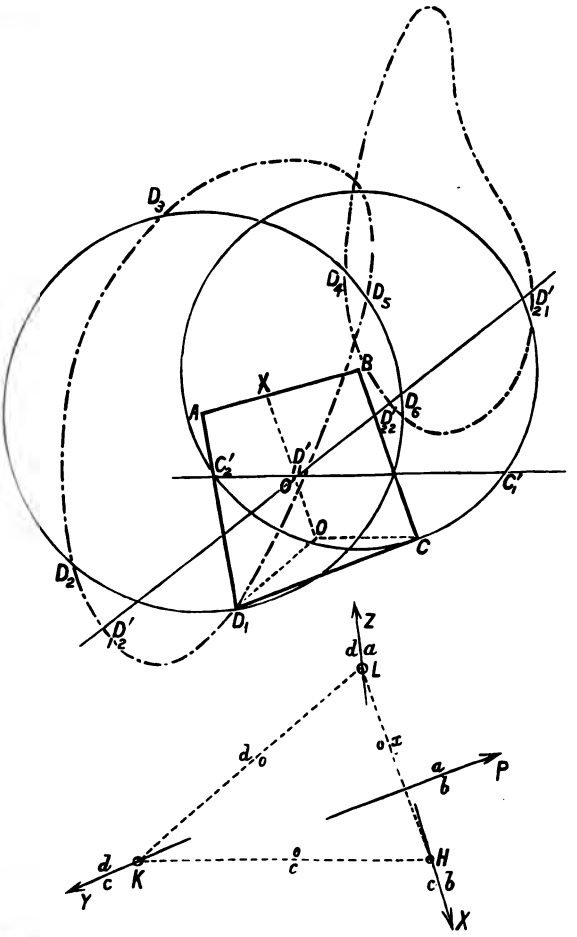
Solution by the PROPOSER.

If the given force is reversed, it forms with the other three a system in equilibrium. The problem therefore is equivalent to the following: — Four co-planar forces P, X, Y, Z are in equilibrium. The force P is represented by a given line AB and acts along a given line *ab*. The forces X, Y, and Z are of given magnitudes and act through given points H, K, and L respectively. It is required to determine the directions of X, Y, and Z.

Suppose that BC, CD, and DA represent X, Y, and Z respectively, so that the lengths of these three lines are given. Divide AB in X, so that



AX/XB = the ratio in which HL is divided by the straight line *ab*. Then the force P is equivalent to AX at L and XB at H. The system of four

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forces is therefore equivalent to DX at L, XC at H, and CD at K, and it is sufficient for equilibrium that straight lines XO, CO, and DO, parallel respectively to LH, HK, and KL, should be concurrent at O. The point X being known, the straight line XO can be drawn parallel to LH, and in it can be taken any point O′. Then O′C′ can be drawn parallel to KH to meet the circle centre B and radius X in the point C′. This gives two possible positions of C′, marked C'_1 and C'_2. Draw O′D′ parallel to LK, taking C′D′ equal to Y. This gives four possible positions of D′, marked ${}_1D'_1$, ${}_2D'_1$, ${}_1D'_2$, and ${}_2D'_2$. By varying the position of O′ we can construct the locus of D′. It is marked — · — · — · — in the figure. The position of the point D is determined by the intersection of this locus with the circle centre A and radius Z. This apparently gives eight possible positions of D, of which in the case taken six are real, marked D_1, D_2, D_3, D_4, D_5, D_6. The figure is then easily completed, and the directions of X, Y, and Z determined.

15843. (Otto Meissner.)—Required to show, by a single trial, that 20857 is a prime.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.; (II.) *by the* Proposer; (III.) *by* H. L. Little.

(I.) It is not quite clear what is meant by "a single trial." The following solution requires very little tentative work. Since

$$N = 20857 = 8 \cdot 3 \cdot 11 \cdot 79 + 1 = 7 \cdot 9 \cdot 331 + 4,$$

therefore (if N be prime) ± 2, ± 3, ± 7, ± 11 are 2-ic residues of N; therefore $\pm 8 \cdot 3 \cdot 7 \cdot 11 = \pm 1848$ is a 2-ic residue of N; therefore N is expressible in form $(t^2 + 1848u^2)$. On trial it is found that

$$N = 65^2 + 1848 \cdot 3^2,$$

and that $u \neq 1, 2$, and cannot be > 3, so that this is the only partition of this kind. Hence N is prime (because 1848 is one of Euler's idoneal numbers, *i.e.*, is such that, if N have only one partition of the kind, it cannot be prime, or a power of a prime).

(II.) $$2^{33} + 1 = (2^{11} + 1)(2^{22} - 2^{11} + 1) = 2049 \cdot 4192257$$
$$= 3 \cdot 683 \cdot 3 \cdot 1397419.$$

The last factor has no other divisors but such of form $66x + 1$, that is 67, 199, ... We have now $1397419 = 67 \cdot 20857$, and therefore 20857 is prime, because it is not divisible by 67 and the next divisor, 199, is $> \sqrt{20857}$, which lies between 144 and 145.

(III.) Factorizing 20857 as in Solution (II.), Question 15827,* we have

$$(10429)^2 - (10428)^2 = 20857, \text{ where } 9d^2 = (10428)^2,$$

or $$20857 \times 1 = 20857;$$

therefore 20857 is prime, a general result obtained without tentative work.

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* For solutions of Question 15827 see p. 90 of this volume.—Ed.

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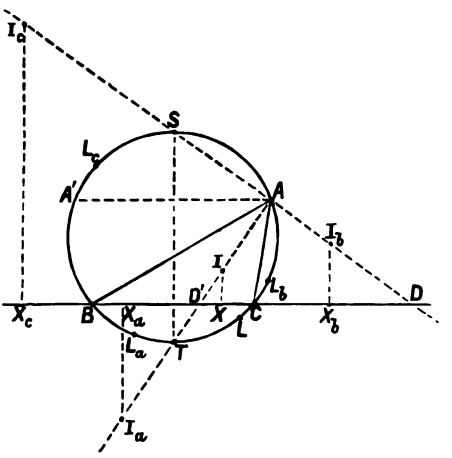
15274. (A. J. Murphy.)—I, I_a, I_b, I_c are the centres of the in- and ex-circles of a triangle ABC, and these circles touch the side BC in X, X_a, X_b, X_c respectively. The internal and external bisectors of the angle A meet the circum-circle in T and S respectively, and SI, SI_a, TI_b, TI_c meet the circum-circle in L, L_a, L_b, L_c respectively. Show that LX, L_aX_a, L_bX_b, and L_cX_c meet on the circum-circle and determine the position of the point of concurrence. Show also that L, L_a, L_b, L_c are the points of contact of the circum-circle with the four circles that can be described to touch the circum-circle and the sides AB, AC.

Solution by C. E. Youngman, M.A.

Transform the figure by symmetric inversion from A, that is, first invert with centre A and radius $\sqrt{(bc)}$, and then reflect the result in the bisector $AITI_a$. This changes B into C, and C into B, and therefore the circle ABC into the line BC; and S into the point D, where AS meets BC. The circle $BICI_a$ will remain unchanged, because it must still go through B and C and cut AT orthogonally; therefore I becomes I_a. The line SIL therefore becomes the circle ADI_a; and this cuts BC again at X_a, because $I_aX_aD = 90° = I_aAD$; therefore X_a is the inverse of L. But BC touches at X_a one of the circles which touch AB, AC, and BC; therefore the circle ABC touches at L one of the circles that touch AC, AB, and the circle ABC. Also AT bisects the angle LAX_a. Similar reasoning holds with X and L_a, X_b and L_c, X_c and L_b.



In the circle ABC draw the chord AA′ parallel to BC. Then the triangles A′CB, ABC are equal, and

$$\angle BA'X_a = \angle CAX = \angle BAL_a = \angle BA'L_a;$$

therefore $A'X_aL_a$ is a straight line; and similarly A′XL, ... Indeed, so long as points x, x' on BC are equidistant from its mid-point, the lines joining each to the inverse of the other must meet at A′. Also the circle Axx' evidently goes through A′; therefore the line ll', joining the inverses of x, x', cuts BC at a fixed point. Further, the circles ABx, ACx' meet again on the median from A; therefore Cl, Bl' meet on the symmedian.

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15842. (C. Bickerdike.)—Three motorists start from three different towns and meet at a junction and, on comparing notes, find that the sum of the distances travelled was a minimum, the distance of the places from each (as the crow flies) being a, b, and c miles respectively. Find the speed of each car, all starting and meeting at the same time, the time from starting to meeting being n hours.

Solution by R. Chartres.

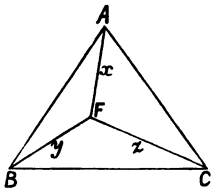
Let $d = x + y + z =$ a minimum;

then F is Fermat's point, and

$$d = \sqrt{[\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\,\Delta]},$$

and $x = (b^2 + c^2 - a^2 + 4\Delta/\sqrt{3})/2d$, ...

Divide by n to get the speed per hour.

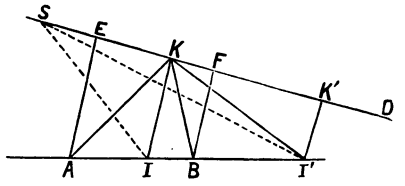


15802. (*Communicated by* James Blaikie, M.A.)—Étant donnés deux points A et B, on prend les points I et I′ qui partagent le segment AB dans un rapport donné α/β et l'on considère toutes les droites D telles que le rapport de leurs distances aux points A et B soit égal à α/β. Démontrer que les plans ID et I′D sont rectangulaires. En conclure:—(1) que les droites D qui passent par un point fixe S engendrent un cône dont les sections circulaires sont perpendiculaires aux droites SI et SI′; (2) que les droites D qui sont situées dans un plan Π enveloppent une conique ayant pour foyers les projections des points I et I′ sur le plan Π. Indiquer ce qui arrive lorsque le point S est situé sur la sphère qui a pour diamètre II′ ou lorsque le plan Π est tangent à cette sphère. (See Guichard, *Compléments de Géométrie.*)

Solution by James Blaikie, M.A.

Let E, K, F, K′ be the feet of the perpendiculars drawn from the points A, I, B, I′ to the straight line D. By hypothesis,

AE/BF = AI/BI
= AI′/BI′ = α/β.



The three straight lines AE, IK, BF are in three parallel planes; therefore KE/KF = IA/IB = AE/BF. The right-angled triangles AEK, BFK are accordingly similar, whence

$$AK/BK = AE/BF = AI/BI = AI'/BI'.$$

This shows that KI, KI′ bisect the angle AKB internally and externally. Hence IK, which is perpendicular to D, is also perpendicular to KI′, and therefore IK is perpendicular to the plane I′D. The plane ID, which

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contains a perpendicular to the plane I'D, is therefore perpendicular to that plane.

(1) SKI, IKI' being right angles, the locus of K is the intersection of the spheres on SI and II' as diameters, that is, a circle whose plane is perpendicular to the line joining the mid-points of SI and II', and therefore also perpendicular to SI'. Similarly the locus of K' is a circle whose plane is perpendicular to SI. Thus the straight line D traces out a cone having two systems of circular sections perpendicular to SI and SI'. If the point S is on the sphere with II' as diameter, this cone reduces to two planes, perpendicular to the plane KII', and passing through KI and KI'.

(2) Let i, i' be the projections of I, I' on the plane Π. Then, as IK is perpendicular to the straight line D in the plane Π, Ki and similarly $K'i'$ are perpendicular to D. And, since the points K, K' describe a circle (the intersection of the plane Π with the sphere on II' as diameter), the straight lines D envelop a conic having i, i' as foci. This conic is an ellipse or a hyperbola according as the plane Π meets the straight line II' between I and I' or not. If the plane Π is tangent to the sphere of diameter II', the conic is a hyperbola reduced to the straight line passing through the point of contact and perpendicular to ii'.

15805. (A. H. Bell.)—Is 9462853 a prime number?

Another Solution by Professor E. B. Escott.

The number may be written $3133^2 - 594^2 = 3727 \cdot 2539$. These factors are prime.

15852. (M. V. A. Sastry, B.A.)—The normals at the points P, Q, R, S of an ellipse pass through the same point O and cut the ellipse again at P', Q', R', S'. If these points be concyclic, show that O lies on a hyperbola which cuts orthogonally all conics passing through the extremities of the axes of the ellipse.

Solutions (I.) *by* W. J. Greenstreet, M.A.; (II.) *by* K. S. Patrachan.

(I.) Let O be (α, β), $P(x_1, y_1)$, $P'(\xi, \eta)$. The ellipse is $x^2/a^2 + y^2/b^2 = 1$.

$$x_1 = (\xi + \lambda\alpha)/(1+\lambda), \quad y_1 = (\eta + \lambda\beta)/(1+\lambda).$$

Eliminating x_1, y_1, and λ from these equations, and from

$$x_1^2/a^2 + y_1^2/b^2 = 1, \quad c^2 x_1 y_1 + b^2\beta x_1 - a^2\alpha y_1 = 0,$$

we obtain the relation

$$2\alpha\beta(x^2 - y^2) + [c^2 - (c^2 + b^2)(\alpha^2/a^2 - \beta^2/b^2)]xy + \ldots = 0,$$

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an equilateral hyperbola through P′, Q′, R′, S′. The general equation of a conic through these points is

$K(b^2x^2+a^2y^2-a^2b^2)+2a\beta(x^2-y^2)+[c^2-(a^2+b^2)(a^2/a^2-\beta^2/b^2)]xy+\ldots=0.$

If this represents a circle, the coefficient of x, y vanishes, and we have

$$(a^2+b^2)(a^2/a^2-\beta^2/b^2)=c^2;$$

so that the locus of O is $b^2x^2-a^2y^2-a^2b^2c^2/(a^2+b^2)$ .....(1),

an equilateral hyperbola with axes parallel to those of the given ellipse, and cutting orthogonally the conics stated.

The Apollonian hyperbola $c^2xy+b^2\beta x-a^2\alpha y=0$ has centre given by

$$c^2y+\beta b^2=0,\quad c^2x-a^2\alpha=0;$$

so that, if P′, Q′, R′, S′ are concyclic, the locus of this centre is a concentric and coaxial ellipse $c^2(a^2+b^2)(b^6x^2-a^6y^2)=a^6b^6$. The equation $c^2xy+b^2\beta x-a^2\alpha y=0$, with the condition $\alpha^2/a^2-\beta^2/b^2=c^2/(a^2+b^2)$, gives for determining the envelope of the Apollonian hyperbola

$$b^2x(d\beta/d\alpha)-a^2y=0,\quad d\beta/d\alpha=b^2\alpha/a^2\beta,\quad b^4x\alpha=a^4y\beta,$$

whence the envelope is the quartic $a^6y^2-b^6x^2=c^2(a^2+b^2)x^2y^2=0$.

This problem appeared some years ago, I think in *Mathesis*, or in the *Revue de Mathématiques Spéciales*.

(II.) (i.) The normal at the point whose eccentric angle is ϕ to the ellipse $x^2/a^2+y^2/b^2=1$ is $ax/\cos\phi-by/\sin\phi=a^2-b^2$. If this meets the curve again at the point ϕ', $a^2\cos\phi'/\cos\phi-b^2\sin\phi'/\sin\phi=a^2-b^2$, or

$$[a^2(\cos\phi'-\cos\phi)]/\cos\phi=[b^2(\sin\phi'-\sin\phi)]/\sin\phi,$$

or $$\tan\tfrac{1}{2}(\phi+\phi')\tan\phi=-b^2/a^2 \quad\ldots\ldots\ldots\ldots(1).$$

If the normal at ϕ passes through the point O $(h,\ k)$ say, then

$$ah/\cos\phi-bk/\sin\phi=a^2-b^2.$$

If $\tan\phi=t$, then $\sin\phi=t/\sqrt{(1+t^2)}$, and $\cos\phi=1/\sqrt{(1+t^2)}$. Hence $[ah\sqrt{(1+t^2)}]/t-bk\sqrt{(1+t^2)}=a^2-b^2$, or $(ah-bkt)^2(1+t^2)=(a^2-b^2)^2$,

or $$a^2h^2t^4-2abhkt^3+t^2(a^2h^2+b^2k^2-a^4e^4)+2abhkt+b^2k^2=0 \quad\ldots\ldots(2),$$

which gives the tangents of the eccentric angles of the points P, Q, R, S. If P′, Q′, R′, S′ be ϕ'_1, ϕ'_2, ϕ'_3, ϕ'_4, since these lie on a circle,

$$\phi'_1+\phi'_2+\phi'_3+\phi'_4=2m\pi.$$

Again, since the normals at P, Q, R, S are concurrent,

$$\phi_1+\phi_2+\phi_3+\phi_4=(2n+1)\pi.$$

Hence $\frac{1}{2}(\phi_1+\phi'_1)+\frac{1}{2}(\phi_2+\phi'_2)+\frac{1}{2}(\phi_3+\phi'_3)+\frac{1}{2}(\phi_4+\phi'_4)=p\pi+\frac{1}{2}\pi$.
Hence $\tan[\Sigma\frac{1}{2}(\phi_1+\phi'_1)]=\infty$. Hence $1-S_2+S_4=0$, where S_2 and S_4 are the sums of the products of the tangents of $\frac{1}{2}(\phi+\phi')$ taken two together and four together. But $\tan\frac{1}{2}(\phi+\phi')=-b^2/(a^2\tan\phi)$ from (1);

therefore $$1-\frac{b^4}{a^4}\Sigma\frac{1}{\tan\phi_1\tan\phi_2}+\frac{b^8}{a^8}\frac{1}{\tan\phi_1\tan\phi_2\tan\phi_3\tan\phi_4}=0 \quad\ldots(3).$$

But, from equation (2),

$\Sigma\tan\phi_1\tan\phi_2=(a^2h^2+b^2k^2-a^4e^4)/(a^2h^2)$ and $\Pi\tan\phi_1=b^2k^2/(a^2h^2)$.
Hence, substituting in (3),

$$b^2k^2/(a^2h^2)-(b^4/a^4)[(a^2h^2+b^2k^2-a^4e^4)(a^2h^2)]+b^8/a^8=0,$$

or $$a^6k^2+b^6h^2=a^2b^2(a^2h^2+b^2k^2-a^4e^4),$$

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or $$b^2h^2(b^4-a^4)+a^2k^2(a^4-b^4)+a^2b^2(a^2-b^2)^2=0,$$

or $$(b^2h^2-a^2k^2)(a^2+b^2)=a^2b^2(a^2-b^2),$$

or $$h^2/a^2-k^2/b^2=(a^2-b^2)/(a^2+b^2),$$

which shows that the locus of O is the hyperbola

$$x^2/a^2-y^2/b^2=(a^2-b^2)/(a^2+b^2).$$

(ii.) Any conic passing through the extremities of the axes of the given ellipse, *i.e.*, through the points $(\pm a, 0)$, $(0, \pm b)$, is given by

$$x^2/a^2+2\lambda xy+y^2/b^2=1.$$

If this and the conic $x^2/a^2-y^2/b^2=(a^2-b^2)/(a^2+b^2)$ intersect in (x', y'), then the equations of the tangents to them at that point are

$$xx'/a^2+\lambda(xy'+x'y)+yy'/b^2=1 \quad \text{and} \quad xx'/a^2-yy'/b^2=(a^2-b^2)/(a^2+b^2).$$

The condition that these two should be perpendicular is

$$(x'/a^2)(x'/a^2+\lambda y')=(y'/b^2)(y'/b^2+\lambda x'),$$

or $$(x'^2/a^4-y'^2/b^4)+\lambda x'y'(1/a^2-1/b^2)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Since the point (x', y') lies on both conics,

$$x'^2/a^2-y'^2/b^2=(a^2-b^2)/(a^2+b^2) \quad \text{and} \quad x'^2/a^2+2\lambda x'y'+y'^2/b^2=1.$$

Hence $$2\lambda x'y'=1-x'^2/a^2-y'^2/b^2.$$

Hence
$$\begin{aligned}&2(x'^2/a^4-y'^2/b^4)+2\lambda x'y'(1/a^2-1/b^2)\\&=2(x'^2/a^4-y'^2/b^4)+(1-x'^2/a^2-y'^2/b^2)(1/a^2-1/b^2)\\&=(x'^2/a^4-y'^2/b^4)+(x'^2-y'^2)/(a^2b^2)+(1/a^2-1/b^2)\\&=(x'^2/a^2-y'^2/b^2)(1/a^2+1/b^2)+1/a^2-1/b^2\\&=(a^2-b^2)(1/a^2+1/b^2)/(a^2+b^2)+1/a^2-1/b^2=0.\end{aligned}$$

Hence relation (A) is satisfied. Therefore the locus of O is a hyperbola which cuts orthogonally all conics passing through the extremities of the axes of the given ellipse.

15848. (R. F. Davis, M.A.)—Solve the equations

$$(x^3-3acx)^2=4a^3c^3, \quad a^3x^6+(b^3-3abc)x^3+c^3=0 \ldots\ldots \text{(i., ii.)}.$$

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by* R. J. Whitaker.

(I.) The first equation may be written $x^6-6acx^4+9a^2c^2x^2-4a^3c^3=0$. The left side breaks up into the factors $(x^2-4ac)(x^2-ac)^2$. Thus the roots are $\pm\sqrt{(ac)}$ and $\pm 2\sqrt{(ac)}$. The second equation may be written

$$a^3x^6+b^3x^3+c^3-3abcx^3=0 \quad \text{or} \quad (ax^2)^3+(bx)^3+c^3-3(ax^2)(bx)c=0.$$

As is well known, this breaks up into the terms

$$(ax^2+bx+c)(ax^2+\omega bx+\omega^2c)(ax^2+\omega^2bx+\omega c),$$

where ω satisfies the equations $\omega^3-1=0$, $\omega^2+\omega+1=0$. The second and third factors may clearly be written in the form

$$a(x/\omega)^2+b(x/\omega)+c, \quad a(x/\omega^2)^2+b(x/\omega^2)+c.$$

If, then, $\alpha=[-b+\sqrt{(b^2-4ac)}]/(2a)$, $\beta=[-b-\sqrt{(b^2-4ac)}]/(2a)$, the roots of $ax^2+bx+c=0$, the roots obtained from the other factors are $\omega\alpha$, $\omega\beta$, $\omega^2\alpha$, $\omega^2\beta$.

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(II.) (i.) becomes $x^6 - 6acx^4 + 9a^2c^2x^2 - 4a^3c^3 = 0$,

i.e., $(x^2 - ac)^2 (x^2 - 4ac) = 0$;

therefore $x = \pm \sqrt{(ac)}$ or $\pm 2\sqrt{(ac)}$.

(ii.) may be written $x^6 - [(3abc - b^3)/a^3]\, x^3 + c^3/a^3 = 0$, *i.e.*,

$$x^6 - (\alpha^3 + \beta^3)\, x^3 + \alpha^3\beta^3 = 0,$$

α, β being the roots of $ax^2 + bx + c = 0$; therefore

$$x = \alpha,\ \omega\alpha,\ \omega^2\alpha,\ \beta,\ \omega\beta,\ \omega^2\beta,$$

ω, ω^2 being the imaginary cube roots of unity.

Note.—If b be substituted for x in (i.), the result is the condition that (ii.) may have pairs of equal roots.

15854. (*The late* R. Tucker, M.A.)—ABC is a triangle; D, E, F are the points of contact of the ex-circles with the sides BC, CA, AB. Find (i.) area of DEF, (ii.) the Brocard angle of DEF. Prove also that the equation to the circle DEF is

$$2\Sigma\alpha\beta\gamma = \Sigma aa \Sigma \left(\frac{a\cos^2 \tfrac{1}{2}A - (b+c)\sin^2 \tfrac{1}{2}A}{s-a}\, \alpha \right).$$

Solution by C. M. Ross *and others.*

Since $BD = s-c$, $CD = s-b$; $CE = s-a$, $AE = s-c$;

$$AF = s-b,\ BF = s-a,$$

the sum of the areas of the triangles DCE, EAF, BFD

$$= \tfrac{1}{2}[(s-a)(s-b)\sin C + (s-b)(s-c)\sin A + (s-c)(s-a)\sin B]$$
$$= \tfrac{1}{2}(\sin C/c)\,[a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b)].$$

(i.) Therefore the area of the triangle DEF

$$= \tfrac{1}{2}ab\sin C - \tfrac{1}{2}(\sin C/c)\,[\Sigma a(s-b)(s-c)]$$
$$= \tfrac{1}{2}ab\sin C\,[1 - (\sin^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}B + \sin^2 \tfrac{1}{2}C)]$$
$$= \tfrac{1}{2}ab\sin C\,(\cos A + \cos B + \cos C - 1)$$
$$= \tfrac{1}{2}ab\sin C\,(2\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)$$
$$= \tfrac{1}{2}ab\sin C\,[2(s-a)(s-b)(s-c)/abc]$$
$$= \Delta\,(2\Delta^2/sabc) = \Delta r/2R,$$

where r, R are in- and circum-radii of ABC.

(ii.) $\cot\omega = (EF^2 + FD^2 + DE^2)/4\Delta_1$ (Nixon's *Trigonometry*, p. 266)

$$= (EA^2 + AF^2 - 2EA \cdot AF\cos A + FB^2 + BD^2$$
$$- 2FB \cdot BD\cos B + DC^2 + CE^2 - 2DC \cdot CE\cos C)/4\Delta_1$$
$$= [\Sigma(s-a)^2 - \Sigma(s-b)(s-c)\cos A]/2\Delta_1$$
$$= [\Sigma(s-a)^2 + \Sigma(s-b)(s-c) - 2\Sigma(s-b)(s-c)\cos^2 \tfrac{1}{2}A]/2\Delta_1$$
$$= [\Sigma(s-a)^2 + \Sigma(s-b)(s-c) - 2\Delta^2(a+b+c)/abc]/2\Delta_1$$
$$= [3s^2 - 2s(a+b+c) + \Sigma a^2 + 3s^2 - 2s(a+b+c) + \Sigma bc$$
$$- 2\Delta^2(a+b+c)/abc]/2\Delta_1$$
$$= [-\tfrac{1}{2}(a+b+c)^2 + \Sigma a^2 + \Sigma bc - 4s\Delta^2/abc]/2\Delta_1$$
$$= (\tfrac{1}{2}\Sigma a^2 - s\Delta/R)\,2\Delta_1 = \Sigma a^2/4\Delta_1 - s\Delta/2R\Delta_1$$
$$= R\Sigma a^2/2r\Delta - s/r,$$

which is the magnitude of the Brocard angle of the triangle DEF.

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(iii.) Take ABC as triangle of reference. The equation of any circle in trilinears is $\Sigma\beta\gamma \sin A + (la + m\beta + n\gamma)\Sigma a \sin A = 0$..... .....(1).
The points D, E, F are

$$[0,\ (s-b)\sin C,\ (s-c)\sin B];\quad [(s-a)\sin C,\ 0,\ (s-c)\sin A];$$
$$[(s-a)\sin B,\ (s-b)\sin A,\ 0].$$

Now (1) passes through D, E, F; therefore

$$(s-b)(s-c)\sin A + a[m(s-b)\sin C + n(s-c)\sin B] = 0 \quad .....(2),$$
$$(s-c)(s-a)\sin B + b[n(s-c)\sin A + l(s-a)\sin C] = 0 \quad .....(3),$$
$$(s-a)(s-b)\sin C + c[l(s-a)\sin B + m(s-b)\sin A] = 0 \quad .....(4).$$

These equations reduce to

$$1 + cm/(s-c) + bn/(s-b) = 0,\quad 1 + an/(s-a) + cl/(s-c) = 0,$$
$$1 + bl/(s-b) + am/(s-a) = 0.$$

From these $2l = [a\sin^2\tfrac{1}{2}A - (b+c)\cos^2\tfrac{1}{2}A + 2(s-a)]/(s-a)$
$= [a(1-\cos^2\tfrac{1}{2}A) - (b+c)(1-\sin^2\tfrac{1}{2}A) + 2(s-a)]/(s-a)$
$= -[a\cos^2\tfrac{1}{2}A - (b+c)\sin^2\tfrac{1}{2}A]/(s-a)$ .....(5).
Similar results follow for m and n. (1) may be written

$$\Sigma a\beta\gamma + \Sigma la \,.\, \Sigma aa = 0;$$

therefore, using (5), it reduces to

$$2\Sigma a\beta\gamma = \Sigma aa\, \Sigma \left(\frac{a\cos^2\tfrac{1}{2}A - (b+c)\sin^2\tfrac{1}{2}A}{s-a}\right) a.$$

15827. (Robert W. D. Christie.) — Factorize 9,702,044,791, and show the principle is general for a certain class of numbers.

Solution (I.) *by the* Proposer; (II.) *by* H. L. Little.

(I.) *A.* Let $N = (10^n x + a)(10^n x + b) = 10^{2n}x^2 + 10^n x(a+b) + ab$.
Assume $a + b = 10^n$; therefore $N = 10^n x[10^n x(x+1)] + ab$ where n, $a+b$, ab are supposed given. In the proposed example we have

$$N = 9702044791;$$

thus $10^n x[10^n x(x+1)] = x^2 + x = 9702,\quad a + b = 10^n = 10^3$ ... (1, 2),
and $ab = 044791$ ..... (3).
(1) is an ordinary quadratic giving $x = 98$; (2) and (3) find $a = 593$, $b = 047$. Consequently the factors are
$10^n x + a = 10^3 . 98 + 953 = 98953$ and $10^n x + b = 10^3 . 98 + 047 = 98047$.
Various aids and devices may be used to shorten the work.

B. Or thus:
$N = (10^n x + a)(10^n x + b) = [10^n \tfrac{1}{2}(a+b)] - [\tfrac{1}{2}(a+b)]^2 = N^1 - \text{remainder}$, and the factors are $[10^n x + \tfrac{1}{2}(a+b)] \pm \tfrac{1}{2}(a-b)$, *i.e.*, sum of square root and root of remainder. Here

$$N = 9702044791 = (10^3 x + a)(10^3 x + b)$$
$$= (10^3 x + 500)^2 - [\tfrac{1}{2}(a-b)]^2 = N^1 - \text{remainder} = 98500^2 - 453^2.$$

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Thus, since $a+b = 1000$ assumed and $\frac{1}{2}(a-b)$ found, we have $a = 593$ and $b = 047$, whence the factors are $10^n x + a = 10^3 . 98 + 953$ and $10^3 . 98 + \ldots + 047$ as before.

(II.) Now there must be at least two factors, one of which is prime. Let $6n \pm 1$ represent the prime factor, and then $6(n+d) \pm 1$ will represent the other factor, *i.e.*, z and $z+6d$ will represent the two factors; therefore $z(z+6d) + 9d^2 = z^2 + 6zd + 9d^2 = 9702044791 + 9d^2 = \square$. Hence it remains to find two squares whose difference is 9702044791. These square numbers are (by the Diophantine analysis) easily shown to be $(98500)^2$ and $(453)^2$. Hence the factors are $(98500+453)(98500-453)$, *i.e.*, the required factors are 98953 and 98047, which are primes. Also one factor at least must have 1 or 3 for the units digit in the above. The rule is of general application.

10693. (Professor LAMPE, LL.D.)—Find all the values of x that satisfy the equation $64 \sin^2 x \sin^2 2x \sin^2 3x = 7$.

Solution by A. H. BELL.

Bringing the factors to the same arc,

$$\sin^2 2x = 4 \sin^2 x \cos^2 x = 4 \sin^2 x (1 - \sin^2 x) \ldots\ldots\ldots (1),$$

$$\sin 3x = 2 \sin x \cos 2x + \sin x = 3 \sin x - 4 \sin^3 x.$$

Squaring, $$\sin^2 3x = 16 \sin^6 x - 24 \sin^4 x + 9 \sin^2 x \ldots\ldots\ldots\ldots (2).$$

The given equation may therefore be written

$$4 . 16 . 64 \sin^{12} x - 4 . 40 . 64 \sin^{10} x + 4 . 33 . 64 \sin^8 x - 4 . 9 . 64 \sin^6 x + 7 = 0 \ldots\ldots\ldots (3).$$

Let $\sin^2 x = y$; then

$$4096y^6 - 10240y^5 + 8448y^4 - 2304y^3 + 0 - 0 + 7 = 0 \ldots\ldots\ldots (4),$$

$\frac{1}{4096}$, $\frac{1}{10240}$, $\frac{1}{8448}$, $\frac{1}{2304}$, $\frac{1}{18}$, $\frac{1}{7}$, 1, is a geometric series.

Putting $y = \frac{1}{4}z$, equation (4) becomes

$$z^6 - 10z^5 + 33z^4 - 36z^3 + 7 = 0 \ldots\ldots\ldots\ldots (5).$$

Since $\frac{1}{4}z = y$, therefore $\frac{1}{2}\sqrt{z} = \sin x$. The law of the signs would indicate three positive roots for equation (5): one is

$$z = 2 . 445 +, \quad \sin x = 0 . 7818 +, \quad \text{arc } x = 51^\circ\ 26'$$

nearly. One or more roots are between $z = 0$ and 1, balance $z = 2$ and 4; no negative roots.

15825. (B. C. WALLIS, F.C.P., B.Sc.Econ.)—What are the general formulæ to express the relationship between the lengths of the sides of a right-angled triangle in terms of one number n? *E.g.*, when the smallest side is an even number of units long the formula $(n+1)^2 = (n-1)^2 + 4n$ holds when n is a perfect square; also when the smallest side is an odd number of units long the formula $n^2 = (n-1)^2 + (2n-1)$ holds when n is

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such that $2n-1$ is a perfect square. These two formulæ enable us to write down a table showing the lengths of the sides when the short sides are the consecutive numbers beginning with 3.

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by* ALEXANDER HOLM, M.A.

(I.) Let x, y, z be the sides of a right-angled triangle, so that $z^2 = x^2+y^2$. The general solutions of this Diophantine are known to be of two kinds.

$x = \frac{1}{2}(t^2-u^2)$, $y = tu$, $z = \frac{1}{2}(t^2+u^2)$
[t, u both odd or both even integers].....(i.),

$x = (t^2-u^2)$, $y = 2tu$, $z = t^2+u^2$ [t, u any integers] .....(ii.).

Dividing through by u^2, and writing $x/u^2 = \xi$, $y/u^2 = \eta$, $z/u^2 = \zeta$, $t/u = n$. Then ξ, η, ζ may be taken to represent the sides, and are given by either $\xi = \frac{1}{2}(n^2-1)$, $\eta = n$, $\zeta = \frac{1}{2}(n^2+1)$ .....(i.),

or $\xi = n^2-1$, $\eta = 2n$, $\zeta = n^2+1$.....(ii.).

(II.) If the sides are x, y, z, then $x^2 = y^2+z^2$; therefore $x^2-y^2 = z^2$ or $(x+y)(x-y) = z^2$; therefore

$$(x+y)/z = z/(x-y) = \text{a rational number} = n;$$

therefore $x+y-nz = 0$ and $-nx+ny+z = 0$; therefore

$$x/(n^2+1) = y/(n^2-1) = z/2n,$$

where n is any rational number.

15740. (S. C. BAGCHI, B.A.)—Supposing that $\phi(\xi_1, \xi_2, \ldots, \xi_n)$ gives the most probable value of a magnitude which is observed to have values $\xi_1, \xi_2, \ldots, \xi_n$ for n different observations successively such that $\xi_2-\xi_1 = \delta_2$, $\xi_3-\xi_1 = \delta_3$, ..., $\xi_n-\xi_1 = \delta_n$, show that a particular solution of the general differential equation for ϕ satisfies

$$n(\phi-\xi_1) = (n+1)\sum_{r=2}^{n}\delta_r.$$

Solution by the PROPOSER.

Adopting the method of Gauss, we represent by $\phi(\delta)\,d\delta$ the probability that an error should lie between δ and $\delta+d\delta$. Hence in the present case the most probable value of ϕ will be that which will render maximum the product $\psi(z-\xi_1)\,\psi(z-\xi_2)\ldots\psi(z-\xi_n)$, if z be the value of the magnitude measured, and $\psi(\delta)$ denotes the function of δ which is proportional to the error δ. Thus after certain easy reductions we get the differential equation

$$\partial\phi/\partial\xi_1 + \partial\phi/\partial\xi_2 + \ldots + \partial\phi/\partial\xi_n = 1,$$

and we see a particular integral of this evidently satisfies

$$n(\phi-\xi_1) = (n+1)\sum_{r=2}^{n}\delta_r,$$

if we take the function in the general solution to denote a summation only.

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15795. (S. C. Bagchi, B.A.)—The picture of a system of latitude and longitude obtained by orthogonal projection on one of the polar-tangent planes is the same as that (on any plane) obtained by inversions from properly chosen centres.

Solution by the Proposer.

The system may be regarded as made up of a sphere, a set of co-axial diametral planes, and a set of parallel planes. Now invert with respect to a sphere whose centre is on the given sphere. We get a plane into which the given sphere inverts, a system of spheres touching at a point into which the parallel planes which determine the circles of latitude invert, and a system of co-axial spheres, these being the inverses of the diametral planes determining the longitudes. Thus for the inverse figure we get a system of co-axial circles together with another system of co-axial circles passing through the limiting points of the former. Now inverting with respect to one of these limiting points, we get a system of concentric circles and concurrent diameters.

2664. (T. Cotterill, M.A.)—(1) Five points (no three in the same line, and no four in the same plane) determine, by the lines and planes through them on a plane, a system of ten points and ten lines, the points lying in threes on the lines and the lines passing in threes through the points (Cayley). Show that the figure is its own polar reciprocal to a conic; and that, if a conic and triangle in its plane are given, the rest of the figure can be constructed. (2) A quadric through the five points cuts the plane in a conic containing triangles conjugate to the fixed conic. There is a quadric passing through the fixed conic, to which one of the five points and the plane are pole and polar, the remaining four points forming a self-conjugate tetrahedron.

Solution by Professor Nanson.

The ten coefficients of the point equation of any quadric through five points 1, 2, 3, 4, 5 are connected by five linear equations, and therefore the six coefficients a, b, c, f, g, h, say, of the point equation of the section of the quadric by a fixed plane $w = 0$ are connected by one linear equation. Denoting this linear equation by

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh = 0,$$

it follows that the section in question is harmonically circumscribed about a fixed conic whose tangential equation is $(ABCFGH)(\xi\eta\zeta)^2 = 0$. Now the plane through the points 1, 2, 3 and an arbitrary plane through the points 4, 5 form a quadric through the five points. If, then, we denote the meets of w with the plane through 1, 2, 3 and the line through 4, 5 by 123, 45 respectively, it follows that the line 123 is conjugate, with respect to the fixed conic, to any line through 45 in the plane w. Hence, then, 123 is the polar of 45 to the fixed conic. Thus, then, the system

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of ten points and ten lines is such that any point or line is the pole or polar of the complementary line or point, *i.e.*, the system is its own polar reciprocal with respect to the fixed conic.

Given the conic and a triangle with vertices 14, 24, 34 and sides 234, 314, 124, the polar triangle with sides 235, 315, 125 and vertices 15, 25, 35 is known. The points 23, 31, 12 are then given as the collinear meets of corresponding sides, and the lines 145, 245, 345 are given as the concurrent joins of corresponding vertices, the line 123 being the axis and the point 45 the pole of the homology.

There is one quadric Q_6 self-conjugate to 1, 2, 3, 4 which has 5 for the pole of the given plane w. The meet of w with the plane through 1, 2, 3 is clearly the polar line of the join of 4, 5 with respect to Q_6. Hence the point 45 has the line 123 for polar with respect to the conic in which Q_6 cuts w. Since the points 15, 25, 35 similarly have given polars in regard to the conic $Q_6 w$, that conic must be identical with the fixed conic.

15793. (Professor Nanson.)—Eliminate x, y, z, x', y', z' from

$$lx + my + nz = 0, \quad lx' + my' + nz' = 0;$$

$$(abcfgh)(xyz)^2 = 0, \quad (abcfgh)(x'y'z')^2 = 0, \quad (abcfgh)(xyz)(x'y'z') = 0.$$

Solution by the Proposer.

The first four equations show that $x : y : z$, $x' : y' : z'$ are the two sets of values of $X : Y : Z$ given by

$$(abcfgh)(XYZ)^2 = 0, \quad lX + mY + nZ = 0 \quad \ldots\ldots\ldots\ldots(1, 2).$$

Now, eliminating X from (1), (2), we get $CY^2 + BZ^2 - 2FYZ = 0$, where A, ..., F, ... are the co-factors of $a, \ldots, f, \ldots$ in the bordered discriminant Δ of (1) and (2). Hence

$$xx' : yy' : zz' : yz' + y'z : zx' + z'x : xy' + x'y = A : B : C : 2F : 2G : 2H;$$

and therefore, from the fifth given equation,

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh = 0.$$

Since the first member is 2Δ, the required result of elimination is $\Delta = 0$, and this is geometrically obvious, since two points of a conic cannot be conjugate unless they coincide.

15818. (R. F. Davis, M.A.)—A triangle circumscribes a given circle (centre I, radius r) and has its orthocentre at a given point H. Prove that the angular points lie on a fixed conic whose focus is at H, and whose directrix KK′ is found by producing IH to X, so that $IH.IX = 2r^2$, and drawing KXK′ perpendicular to IHX. [Amplification of No. 1114 of Wolstenholme's Problems.]

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Solution by Professor SANJÂNA, M.A.

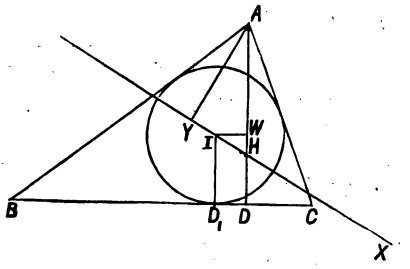
Draw AY perpendicular to IH and IW to AD; HD, ID_1 perpendiculars to BC. It is well known that

$IH^2 + AH \cdot HD = 2r^2$;

hence in the present case

$AH \cdot HD = IH \cdot IX - IH^2$

$= IH \cdot HX$...(a).



Now

$$\frac{YX}{AH} = \frac{YH}{AH} + \frac{HX}{AH}$$

$$= \frac{WH}{HI} + \frac{HD}{HI} \text{ [by (a)]} = \frac{ID_1}{HI}.$$

Hence $AH : YX = \delta : r$; so that the locus of A is a conic, focus H and directrix perpendicular to IH at X. This conic is an ellipse, parabola, or hyperbola, according as H is within, upon, or without the in-circle. [See Wolstenholme's problem above mentioned.]

15840. (R. CHARTRES.)—Find the mean distance of all points within a square from one of the corners in terms of a parabolic arc. (Elementary proof.)

Solution by the PROPOSER.

Let A be the centroid of the triangle LMN, and the vertex of the parabola AFL.

$ST = SL = SG$.

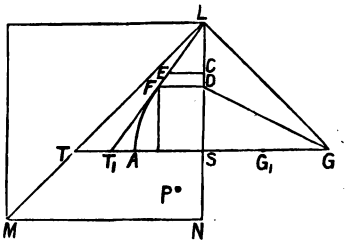
By similar triangles ultimately

$EF = (CD \cdot DG)/SG$

$= (CD \cdot DG)/SL$;

therefore the arc AFL or

$$\Sigma(EF) = \frac{\Sigma(CD \cdot DG)}{SL}$$



= the mean distance of all points in SL from G,

= " " " ST from L,

= " P, or all points in the square from L.

A similar proof will hold for a rectangle, &c.

15849. (Professor NEUBERG.)—On sait que les pieds M, M', M'', M''' des normales abaissées d'un point P sur une conique Σ sont sur une hyperbole équilatère Σ' passant par P et par le centre C de Σ. Étant donnés l'hyperbole équilatère $\Sigma'(xy = 1)$ et les trois points M, M', M'' sur cette

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courbe, trouver (1) la relation entre les abscisses des points M, M′, M″, M‴ ; (2) la relation entre les abscisses des points M, M′, M″, P ; (3) celle entre les abscisses des points M, M′, M″, C.

Solution by C. M. Ross *and* K. S. Patrachan.

Let $(l, 1/l)$, $(m, 1/m)$, $(n, 1/n)$, $(\lambda, 1/\lambda)$, $(h, 1/h)$, $(p, 1/p)$ be the points M, M′, M″, M‴, P, and C which are on

$$xy = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of the conic Σ whose centre is C is of the form

$$Ap^2x^2 + Bp^2y^2 - 2Axp^3 - 2Bpy + Ap^4 + B - p^2 = 0 \quad \ldots\ldots\ldots\ldots(2).$$

The abscissæ of M, M′, M″, M‴ satisfy the equation obtained by eliminating y between (1) and (2). From (1), $y = 1/x$; therefore (2) becomes

$$Ap^2x^4 - 2Ap^3x^3 + x^2(Ap^4 - p^2 + B) - 2Bpx + Bp^2 = 0 ;$$

therefore $\Sigma l = 2p$, $\Sigma lm = (Ap^4 - p^2 + B)/Ap^2$, $\Sigma lmn = 2B/Ap \ldots(3, 4, 5)$,

$$lmn\lambda = B/A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Again, the equation of the conic Σ referred to C as centre is $Ax^2 + By^2 = 1$, and the co-ordinates of the feet of the four normals drawn from P lie on the rectangular hyperbola

$$hp(A - B)xy + hpBy(h - p) + Ax(h - p) = 0.$$

Referring this equation to its centre, it becomes

$$hp(A - B)xy = AB(h - p)^2/(A - B).$$

This is identical with $\Sigma'(xy = 1)$; therefore

$$AB(h - p)^2 = hp(A - B)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

(1) From (5) and (6), $\Sigma lmn/lmn\lambda = 2/p = 4/\Sigma l$ by (3) ; therefore

$$(1/l + 1/m + 1/n + 1/\lambda)(l + m + n + \lambda) = 4,$$

the relation connecting the abscissæ of M, M′, M″, M‴.

(2) Put $l + m + n = D$, $lm + mn + nl = E$, $lmn = F$; then (5) is $F + E(2p - D) = 2B/Ap$, by (3) ; (6) is $F(2p - D) = B/A$, by (3). From these last equations $Fp + Ep(2p - D) = 2F(2p - D)$; therefore

$$p = \{ED + 3F \pm \sqrt{[(ED - 9F)(ED - F)]}\}/4E \equiv L/4E.$$

Again, put $B/A = t$; then

$$t = F\{3F - ED \pm \sqrt{[(ED - 9F)(ED - F)]}\}/2E \equiv FM/2E.$$

Substituting this value for B/A in (7),

$$(16E^2h^2 + L^2)/2hL = (4E^2 + F^2M^2)/FM ;$$

therefore $8E^2h(2hFM - L) = FLM(2hFM - L)$;

therefore $2hFM = L$ or $8E^2h = FLM$.

Considering the first relation just found, it reduces to

$$2hF(3F - ED) - ED - 3F = \pm\sqrt{[(ED - 9F)(ED - F)]}(1 - 2hF).$$

Squaring and reducing, $2h^2F^2 - 5hF + hED + 2 = 0$,

or $2h^2l^2m^2n^2 - 5hlmn + 2 + h[l^2(m + n) + m^2(n + l) + n^2(l + m)] + 3hlmn = 0$;

therefore $2h^2l^2m^2n^2 + h[\Sigma l^2(m + n) - 2lmn] + 2 = 0$,

the relation between the abscissæ of M, M′, M″, P.

(3) Again $2p^2E - p(ED + 3F) + 2FD = 0$,

i.e., $2p^2(lm + mn + nl) - p[\Sigma l^2(m + n) + 6lmn] + 2lmn(l + m + n) = 0$.

Dividing by lmn, it reduces to

$$2p^2(1/l + 1/m + 1/n) - p[(l + m + n)(1/l + 1/m + 1/n) + 3] + 2(l + m + n) = 0,$$

the relation between the abscissæ of M, M′, M″, C.

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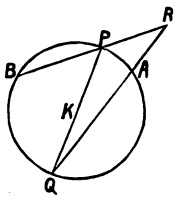
15871. (Professor SANJÂNA, M.A. Suggested by Question 15748.)—A, B, C are three fixed points on a conic, and from another fixed point K a chord is drawn terminated by the curve at P and Q. Find the locus of the intersection of AQ and BP, and investigate when it is of the first degree.

Solution by W. J. DOBBS, M.A.

The point C is not required in the Question. Project the conic into a circle of centre K. Then, if BP and AQ meet at R,

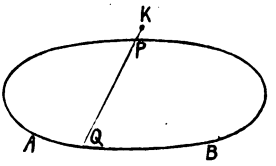
$$\angle BRA = \angle BPA - 90° = \text{const.}$$



Therefore the locus of R is a circle through A and B, which reduces to the straight line at infinity when BA is a diameter. Hence, returning to the original problem, we see that the locus required is a conic through A and B and the intersections of the polar of K with the original conic; and that it reduces to the polar of K when A, B, and K are collinear.

Mr. H. L. TRACHTENBERG, B.A., solves the first part of the Question by isogonal transformation as follows:—

Transform with regard to the triangle KAB. The conic, passing through AB, becomes another conic through AB, and, if P′, Q′ be the points corresponding to P, Q, the straight lines KPQ, AQ, BP become, since they pass through an angular point, the straight lines KP′Q′, AQ′, BP′, passing through the same angular points respectively.



Hence the locus of the intersection of AQ and BP transforms into the locus of the intersection of AQ′ and BP′, a locus which bears the same relation to the transformed conic and the points K, A, B as the locus sought does to the original conic and the points K, A, B. Hence, if the degree of the locus sought is n, that of its transformation is n.

Now a conic through KAB cuts the locus sought in $2n$ points, two of which coincide with A, B, since these are points on the locus found by taking limiting positions KA, KB for KPQ. Hence the $2n-2$ points which do not coincide with an angular point must correspond to the points in which the straight line corresponding to the conic through KAB cuts the transformed locus, but the number of these points is n. Therefore $2n-2 = n$. Hence $n = 2$. Thus the locus required is a conic through A, B.

15862. (F. N. MAYERS, M.A.)—Discuss the possibility or otherwise of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$.

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Note by Lt.-Col. ALLAN CUNNINGHAM, R.E.

In the *Comptes Rendus des Séances de l'Acad. des Sciences*, Paris, t. LXXXIII., p. 1288, there are given the enunciations of several theorems by M. M. F. Proth, one of which is that "If $2^p-2 \equiv 0 \pmod{p}$ a prime), then $2^p-2 \not\equiv 0 \pmod{p^2}$ or p^3)." It is not stated whether this is a mere empirical theorem, or whether the author of the paper had found actual proof of the same.

15679. (ALEXANDER HOLM, M.A.) — In Diophantus v. 19 it is required to find three numbers such that, if each is subtracted from the cube of their sum, the remainder may be a cube.

Diophantus' method leads to the numbers

$$\frac{68574961}{549353259}, \quad \frac{134953109}{549353259}, \quad \frac{162707336}{549353259}.$$

In the *Ladies' Diary* for 1716–17 (Leybourn's *Math. Questions*, Vol. I., pp. 52–62) it is stated that the lowest numbers hitherto found to satisfy the problem are $\frac{13851}{85184}, \frac{18954}{85184}, \frac{19467}{85184}$; required to find still lower numbers, for example, $\frac{351}{3136}, \frac{832}{3136}, \frac{833}{3136}$.

Solution by the PROPOSER.

Let x, y, z be the required numbers, and their sum

$$x+y+z = u \quad \ldots\ldots(1);$$

then
$$\left.\begin{array}{l} u^3-x = \text{a cube} = (lu)^3, \quad u^3-y = \text{a cube} = (mu)^3 \\ u^3-z = \text{a cube} = (nu)^3 \end{array}\right\} \quad \ldots\ldots(2),$$

where l, m, n are arbitrary; therefore

$$x = (1-l^3)\,u^3, \quad y = (1-m^3)\,u^3, \quad z = (1-n^3)\,u^3 \quad \ldots\ldots(3).$$

Hence, from (1), $3-l^3-m^3-n^3 = 1/u^3 = \square$. Thus it is necessary to seek for three positive cubes each < 1 [*cf.* (2), (3)], such that, their sum being taken from 3, a square is left. Setting out from Diophantus' particular case, $3-(\frac{2}{3})^3-(\frac{4}{3})^3-(-\frac{1}{3})^3 = \square = (\frac{3}{3})^2$, let us take $\frac{2}{3}$, $\frac{4}{3}-\mathrm{X}$, $-\frac{1}{3}+\mathrm{X}$ for the roots of the required cubes. These assumptions leave the sum of the roots of the cubes unaltered. (Diophantus keeps the sum of the cubes themselves constant.) Thus $3-(\frac{2}{3})^3-(\frac{4}{3}-\mathrm{X})^3-(-\frac{1}{3}+\mathrm{X})^3 = \square$; therefore $\frac{2}{3}+\frac{7}{15}\mathrm{X}-\frac{1}{3}\mathrm{X}^2 = \square = (-\frac{2}{3}+r\mathrm{X})^2$, where r is arbitrary; therefore $\mathrm{X} = (36r+7)/(12r^2+6)$. To make the last two cubes positive, choose

$$\mathrm{X} = (36r+7)/(12r^2+6) > \tfrac{1}{3} < \tfrac{4}{3};$$

whence $r > \frac{1}{3}(9-\sqrt{87}) < \frac{1}{4}(9-\sqrt{87})$ or $> \frac{1}{4}(9+\sqrt{87}) < \frac{1}{3}(9+\sqrt{87})$. If $r = \frac{1}{15}$, then $\mathrm{X} = \frac{8}{15}$. Thus $3-(\frac{1}{15}\frac{2}{5})^3-(\frac{3}{15})^3-(\frac{1}{15})^3 = \square = (\frac{2}{15}\frac{3}{5})^2$. Taking $l = \frac{1}{15}\frac{2}{5}$, $m = \frac{3}{15}$, $n = \frac{1}{15}$, $u = \frac{1}{2}\frac{5}{5}$, we have

$$x = \tfrac{3}{3}\tfrac{5}{1}\tfrac{1}{5}\tfrac{7}{5}, \quad y = \tfrac{3}{3}\tfrac{3}{1}\tfrac{3}{5}\tfrac{3}{5}, \quad z = \tfrac{3}{3}\tfrac{3}{1}\tfrac{3}{5}\tfrac{3}{5}.$$

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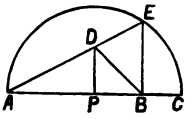
15844. (Robt. W. D. Christie.)—Divide a straight line into two such parts that the square on one part shall equal three times the square on the other. Can it be solved generally?

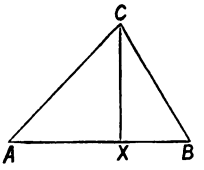
Solutions (I.) *by* Henry Riddell, M.E.; (II.) *by* Ram Krishna Rao; (III.) *by* W. J. Greenstreet, M.A., *the* Proposer, *and others*; (IV.) *by* Lt.-Col. Allan Cunningham, R.E.; (V.) *by* Professor Langhorne Orchard, M.A., B.Sc.

(I.) Let it be required to divide AB into two parts AP, PB, so that $AP^2/PB^2 = m/n$. Produce AB to C, making $AB/BC = m/n$, which can always be done as long as the ratio m/n can be got in linear form. On AC describe a semicircle, and erect BE perpendicular to AB and cutting the semicircle at E. Join AE, and through B draw BD, making the angle ABD half a right angle, and drop DP perpendicular to AB. Then



$$AP^2/PB^2 = AP^2/PD^2 = AB^2/BE^2 = AB/BC = m/n.$$

(II.) Let AB be the given straight line. At A and B make angles of 45° and 60° respectively. Draw CX perpendicular to AB. Then X shall be the required point. Now $CX = AX$ and $CX/BX = \sqrt{3}$; therefore $AX/BX = \sqrt{3}$; therefore $AX^2 = 3BX^2$; therefore X is the required point.



The general solution for dividing a straight line into two parts so that the square on one part may be n times the square of the other:—Take a straight line PQ. Divide it at R in the ratio of 1 to n. Find a mean proportional PS to PR and RQ.

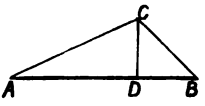
$$PS^2 = PR.RQ = nPR^2.$$

Now divide AB at X in the ratio of PS to PR.

Then $AX : XB = PS : PR$;

therefore $AX^2 : XB^2 = PS^2 : PR^2$; therefore $AX^2 = nXB^2$.

(III.) Let AB be the given line. Construct a triangle ABC with angles A = 30°, B = 45°. Then $CD = \frac{1}{2}AC$, CD = DB. Then CD, perpendicular on AB, gives D, the point required. For $4CD^2 = AC^2$; therefore $AD^2 + CD^2 = 4CD^2$; therefore $AD^2 = 3CD^2 = 3DB^2$. If $AB = a$ and D is on AB such that $AD^2 = nDB^2$, let $AD = x$; then $x^2 = n(x-a)^2$,



$x^2(n-1) - 2anx + a^2n = 0$; therefore $x = a\sqrt{n}/(\sqrt{n} \pm 1)$. Now a length $a\sqrt{n}$ can always be drawn and the section required can always be effected.

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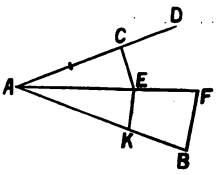
Perhaps the simplest process is as follows:—

(1) $n = 5$.—Trisect any line AD in C; draw CE perpendicular to CD and equal to it. Produce AE to F, so that CE = EF. Join FB. EK, parallel to FB, gives the point required. For



$$AE^2 = AC^2 + CE^2 = 5CE^2, \quad EF^2 = CE^2,$$
$$AE/EF = AK/KB,$$
$$AK^2/KB^2 = AE^2/EF^2 = 5.$$

(2) $n = 6$.—Divide any line AD in C, so that $AC^2 = 5CD^2$. Draw CE perpendicular and equal to CD. Produce AE to F, so that

$$EF = CE = CD.$$

Draw EK parallel to FB. Then

$$AE^2 = AC^2 + CE^2 = 6CE^2, \quad EF^2 = CE^2, \quad AK^2/KB^2 = AE^2/EF^2 = 6,$$

and, by repeating this process, we get each point of section in turn.

(IV.) In the general case (when n is an integer) all that is necessary is to find two lines in the ratio of $1 : \sqrt{n}$. Let $n = m^2 + \mu$, where m^2 is the integer square next $< n$. Take a line AB = m units, and draw BC = 1 unit perpendicular to AB. Then $AC = \sqrt{(m^2+1)}$ units. Similarly, lines AD, AE, ... $= \sqrt{(m^2+2)}$, $\sqrt{(m^2+3)}$, ... units can be found in succession, until a line, say AP, is found $= \sqrt{(m^2+\mu)}$ units. If m be large, this process can be much curtailed by taking $\mu = m_1^2 + \mu_1$, where m_1^2 is the integer square next $< \mu$, and taking BC = m_1 units (instead of 1 unit) in preceding process. If μ_1 be still large, it can be resolved into $\mu_1 = \mu_2^2 + \mu_2$, and so on. The last line AP will be of $\sqrt{(m^2+\mu)} = \sqrt{n}$ units. Produce this line AP to Q, taking PQ = 1 unit. Then AQ will be a line divided at P in the required ratio $\sqrt{n} : 1$. It only remains to divide the given line in the same ratio.

Similarly a straight line could be divided in ratio $\sqrt{m} : \sqrt{n}$ (when m, n are integers).

(V.) Let AB = a be the straight line, x and $(a-x)$ the two parts. Then, for the general case,

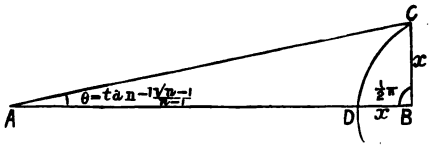
$$(a-x)^2 = nx^2;$$

therefore

$$x/a = (\sqrt{n}-1)/(n-1);$$

or, in the particular case given,

$$x/a = \tfrac{1}{2}(\sqrt{3}-1).$$



Hence, geometrically, as above, where the part DB = x = perpendicular BC. Since the fraction $(\sqrt{n}-1)/(n-1)$ must be positive and $>$ unity, θ is $> \frac{1}{4}\pi$.

15856. (Professor Cochez.)—Trouver le lieu des points tels que leurs polaires relatives à trois cercles concourent en un même point.

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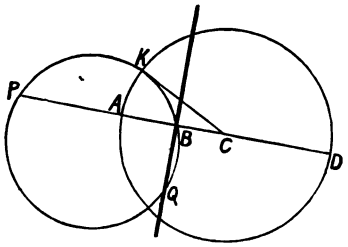
Solutions (I.) *by* HENRY RIDDELL, M.E., *and* W. J. GREENSTREET, M.A.;
(II.) *by* H. M. TAYLOR, M.A., F.R.S., *and* C. M. ROSS.

(I.) Let there be any two circles cutting each other at right angles, and through any point P on the circumference of one draw the chord PABD, passing through the centre of the second circle and cutting the two circumferences in P, B, A, D. Since the two circles cut at right angles,



$$CB \cdot CP = CA^2,$$

and hence the polar of P with regard to the circle having C as centre passes through B, and, being at right angles to PC, also passes through Q, the point diametrically opposite P. Hence, if any number of circles be described cutting the first circle at right angles, the polars of P with regard to them will be concurrent at Q. In general, a circle can be drawn to cut three circles at right angles, and this circle is the required locus.

The case in which the circles are co-axial is worth notice. As a circle cutting a system of co-axial circles at right angles can be drawn through any point, the polars of any point with regard to such co-axial system will concur.

(II.) Let the equations of the three circles referred to rectangular co-ordinates be

$$x^2+y^2-2a_1x-2b_1y+c_1=0,$$
$$x^2+y^2-2a_2x-2b_2y+c_2=0,$$
$$x^2+y^2-2a_3x-2b_3y+c_3=0.$$

Then the condition that the polars of the point (x, y) with respect to the three circles should meet in a point is

$$\begin{vmatrix} x-a_1, & y-b_1, & a_1x+b_1y-c_1 \\ x-a_2, & y-b_2, & a_2x+b_2y-c_2 \\ x-a_3, & y-b_3, & a_3x+b_3y-c_3 \end{vmatrix} = 0.$$

The locus is therefore apparently a cubic curve; but it is easily seen that the coefficients of the highest terms x^2y, xy^2 vanish. This suggests that the line at infinity is part of the locus, as is clear from the fact that the polars of a point at infinity are parallel. The remaining part of the locus is therefore a conic. The coefficient of the term xy is zero, and the coefficients of x^2, y^2 are equal; the locus is therefore a circle.

If we choose the radical centre of the three given circles as origin of co-ordinates, then $c_1 = c_2 = c_3 = t^2$. The coefficients of the terms x, y will then vanish, and the equation of the locus circle reduces to $x^2+y^2=t^2$. This shows that the locus circle cuts the given circles orthogonally. It may be proved that the polars of any point on the locus circle meet in the other extremity of the diameter through that point of the locus circle, and it may be observed that each of the points where the locus circle cuts the line joining the centres of two of the given circles has the same polar with respect to each of those circles. It is

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easily seen that, if the centres of the given circles are collinear, the locus circle reduces to the line at infinity and the line of the centres.

These results can very easily be verified geometrically.

10693. (Professor LAMPE, LL.D.)—Find all the values of x that satisfy the equation $64 \sin^2 x \sin^2 2x \sin^2 3x = 7$.

Solution by the PROPOSER.

The relation $\sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$

$= \sin x (7 - 56 \sin^2 x + 112 \sin^4 x - 64 \sin^6 x)$

leads immediately to the conclusion that the equation

$$64y^6 - 112y^4 + 56y^2 - 7 = 0 \quad \ldots\ldots(1)$$

has the roots $y_1^2 = \sin^2 \frac{1}{7}\pi$, $y_2^2 = \sin^2 \frac{2}{7}\pi$, $y_3^2 = \sin^2 \frac{3}{7}\pi$, and thence

$$\sin^2 \tfrac{1}{7}\pi \sin^2 \tfrac{2}{7}\pi \sin^2 \tfrac{3}{7}\pi = \tfrac{7}{64} \quad \ldots\ldots(2).$$

Comparing (2) with $64 \sin^2 x \sin^2 2x \sin^2 3x = 7 \quad \ldots\ldots(3)$,

we see that $x_1 = \frac{1}{7}\pi$, $x_2 = \frac{2}{7}\pi$, $x_3 = \frac{3}{7}\pi$ are roots of (3). But it is possible that there are still other roots of (3). Reduced to powers of $\sin x = y$, (3) becomes $4096y^{12} - 10240y^{10} + 8448y^8 - 2304y^6 + 7 = 0 \quad \ldots\ldots(4)$.

The polynome forming the left side of (4) must admit of the polynome in (1) as a factor. Dividing, we get

$$64y^6 - 48y^4 - 8y^2 - 1 = 0 \quad \ldots\ldots(5).$$

This equation possesses a positive root for y^2, $= 0{\cdot}9068412 = \sin^2 x$, whence we derive $x = 72° 13' 43{\cdot}2''$. Thus we have the following values for $0 < x < 90°$:—

$$x_1 = \tfrac{1}{7}\pi = 25° 42' 51{\cdot}43'', \qquad x_2 = \tfrac{2}{7}\pi = 51° 25' 42{\cdot}86'',$$
$$x_3 = 72° 13' 43{\cdot}2'', \qquad x_4 = \tfrac{3}{7}\pi = 77° 8' 34{\cdot}29''.$$

Similar questions result from the relations

$$2^{\frac{1}{2}(n-1)} \sin \pi/n \sin 2\pi/n \sin 3\pi/n \ldots \sin [(n-1)\pi]/2n = \sqrt{n},$$
$$2^{\frac{1}{2}(n-1)} \cos \pi/n \cos 2\pi/n \cos 3\pi/n \ldots \cos [(n-1)\pi]/2n = 1,$$

n being any odd number.

15764. (R. F. DAVIS, M.A.)—The tangent at A to the circum-circle of the triangle ABC meets the base BC produced in T. From T perpendiculars are drawn upon AB, AC, meeting the circum-diameter through A in P, Q respectively. Prove (geometrically) that P, Q are equidistant from the circum-centre.

[Want of space prevents the publication of more than a few of the many solutions sent in.—ED.]

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Note by CONSTANCE I. MARKS, B.A.

The problem appears to require something more to complete its solution than has been given in either of the articles sent in. It is to be observed that P and Q will be within or without the circle according as the circum-diameter through A falls without or within the angle A of the triangle ABC. The applicability to both figures of a method of solution based on either one of them is not obvious.

Solutions (I.) *by* C. E. WILLIAMS, M.A.; (II.) *by* J. A. H. JOHNSTON, M.A., *and others;* (III.) *by* A. M. NESBITT, M.A.; (IV.) *by* H. W. WEBSTER, M.D.

(I.) Since $\angle ABA' = 90° = \angle AHP$,

$$A'P : BH = AA' : AB.$$

Also $\angle QTA = 90° - \angle TAC = \angle A'AC$

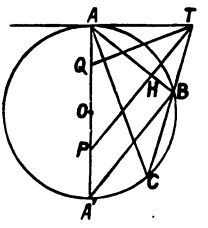
$$= \angle A'BC = \angle PTB.$$

and $\angle HTA = 90° - \angle TAH = \angle BAA'$.

Therefore

$$AQ : BH = AT : TH = AA' : AB.$$

Thus $A'P = AQ$.

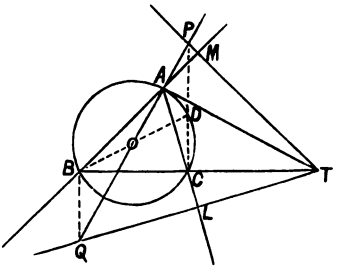


(II.) Let TQ meet AC in L and AB in M. Join PC, QB.

Then $TA^2 = TC \cdot TB$.

But $TA^2 = TL \cdot TQ$;

for QAT and ALT are right angles. Therefore B, Q, L, C are concyclic; therefore QBT is a right angle. Similarly B, P, M, C are concyclic; therefore PCT is a right angle. Let PC meet the circle in D, and draw diameter BD. Comparison of the triangles OBQ, ODP, which have two angles and a side equal in each, shows at once that OP = OQ.



(III.) Since T, M, A, N are concyclic, the triangles MTN, BAC are similar; therefore $TM/TN = AB/AC$.

And, since

NTA = CAE

(each is the complement of NAT)

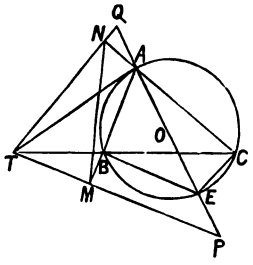
= CBE = BTM,

we have $TM/TN = BM/AN$.

Thus $AB/BM = AC/AN$,

or $AE/EP = AE/AQ$.

Hence AQ = EP and OQ = OP.



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(IV.) TA being a tangent at A,

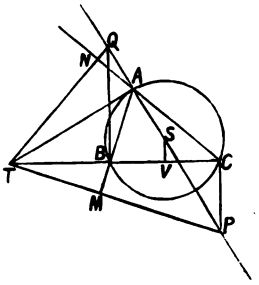
$$\angle TAB = \angle TCA;$$

also $$\angle TAB = \angle TPA,$$

each being the complement of $\angle ATM$; therefore A, C, P, T are cyclic points; therefore TCP is a right angle. Also $\angle TAC = \angle TBA$, as TA is a tangent; $\angle TAC$ = supplement of $\angle TAN$ or of $\angle TQA$; therefore T, B, A, Q are cyclic points; therefore

$$\angle TBQ = \angle TAQ,*$$

a right angle; therefore QB, PC, SV are parallel. V bisects BC; therefore S bisects PQ.



15841. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Jack and Jill run a race of 300 yds. to the top of a hill, and then the same distance down the other side to a fixed goal. Going up the hill Jill takes 5 steps to Jack's 4, and going down takes 6 steps to Jack's 5. If Jack's speed in ascending is half what it is in descending, and if the length of Jill's step to Jack's is as 3 to 4, how much start should be given to Jill in order that they may reach the goal together?

Solutions (I.) *by* James Blaikie, M.A.; (II.) *by* B. C. Wallis, F.C.P., B.Sc. (Econ.); (III.) *by* R. J. Whitaker; (IV.) *by* T. Brill.

(I.) If the length of Jill's step be $3a$ and Jack's $4a$, then up hill Jill travels $15a$ while Jack travels $16a$, and down hill Jill travels $18a$ while Jack travels $20a$. Then, if x be the required start and we take Jack's rate up hill as unity, equating the times taken, we have

$$300 + \tfrac{2}{3}\tfrac{5}{4}x = (300 - x)/\tfrac{1}{1}\tfrac{5}{6} + 300/\tfrac{2}{3},$$

which gives $x = 34\frac{2}{3}$ yds., Jill's start.

(II.) Up hill Jill goes 15 units to Jack's 16. Down hill Jill goes 18 units to Jack's 20. Jill goes 300 yds. down hill, while Jack could go $\frac{1}{2}\frac{0}{0}$. 300 = $333\frac{1}{3}$ yds., an excess of $33\frac{1}{3}$ yds. Therefore Jack is $16\frac{2}{3}$ yds. from the top when Jill is just there. Therefore Jack goes $(300 - 16\frac{2}{3}) = 283\frac{1}{3}$ yds. up hill while Jill ascends completely from her starting point. Therefore Jill's distance up is $\frac{1}{1}\frac{5}{6}$ of $283\frac{1}{3} = 265\frac{5}{8}$ yds. Therefore Jill should receive $34\frac{2}{3}$ yds. start.

(III.) Up hill Jack's speed : Jill's speed = 16 : 15; down hill Jack's speed : Jill's speed = 20 : 18. If Jack's speeds are $16x$ and $20y$ yds. per second, then $32x = 20y$. Therefore $y = \frac{8}{5}x$; so that Jack's total time is $300/16x + 300/32x$ seconds, *i.e.*, $900/32x$ seconds. Jill's time down hill

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* Dr. Webster points out that his solution may be adapted to the second case as follows:—When diameter falls outside angle A, $\angle TAC = \angle TAN$ or $\angle TQA$; and $\angle TBQ$ = supplement of $\angle TAQ$, a right angle; &c.

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$= 300/18y$, *i.e.*, $300/48x$ seconds. Therefore Jill has $425/24x$ seconds left for the up hill part of the course. In this time at a speed of $15x$, she covers $265\frac{5}{8}$ yds. She should therefore receive $34\frac{3}{8}$ yds. start.

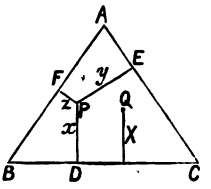
(IV.) In order to calculate the start Jill should receive, we may reckon backwards, assuming that they start together at the goal, their speed in ascending being double that in descending. The distance Jill is from the original starting point when Jack reaches it will give us the start she should receive.

When running from the goal to the top of the hill, Jack's speed is to Jill's speed as (4 . 5) : (3 . 6) or 10 : 9; therefore when Jack reaches the top, *i.e.*, has run 300 yds., Jill has run $300 \cdot \frac{9}{10}$ yds. = 270 yds. While Jill runs the next 30 yds. (300 − 270) at the same speed, Jack begins to run down hill at half his former speed; therefore he runs $30 \cdot \frac{5}{8}$ yds. = $16\frac{7}{8}$ yds. Jill is now at the top of the hill and runs at half her former speed. From data, their relative speeds are as (4 . 4) : (5 . 3) or 16 : 15 (Jack's to Jill's). Jack has to run $300 - 16\frac{7}{8}$ yds. = $283\frac{1}{8}$ yds. till he reaches the starting point. During this time Jill runs $283\frac{1}{8} \cdot \frac{15}{16}$ yds. = $265\frac{5}{8}$ yds. She is then $34\frac{3}{8}$ yds. from the starting point. Therefore she should receive $34\frac{3}{8}$ yds. start.

Note on Two Isogonal Conjugates.

By R. Chartres.

Let P and Q be isogonal conjugates; then, if $x^2 + y^2 + z^2$ be a minimum .....(1), while $ax + by + cz$ = constant = 2Δ .....(2), then $(a^2 + b^2 + c^2)$ times (1) − (2) squared or $(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2$ will be a minimum; therefore $x : y : z = a : b : c$. But $xX = yY = zZ$; therefore $aX = bY = cZ$, or Q is the centroid and P the Lemoine point, and evidently P is the centroid of DEF.



15079. (Rev. T. Roach, M.A.)—Prove geometrically

$$\sin^{-1}\tfrac{4}{5} + \sin^{-1}\tfrac{5}{13} + \sin^{-1}\tfrac{16}{65} = \tfrac{1}{4}\pi.$$

Solution by C. E. Youngman, M.A.

A general formula of this kind can be obtained as follows:—If N be an even number greater than 2, N^2 can always be resolved in two ways into a pair of unequal factors, say $N^2 = ab = cd$, where $a > b$, $c > d$, and $a + b > c + d$. Make two right-angled triangles ABC, ADC, with the right angle C common, and AC = 2N, AB = $a + b$, AD = $c + d$. Then

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BC $= a-b$ and DC $= c-d$. Let p be the perpendicular from D on AB. We have $p \cdot AB = AC \cdot BD$; *i.e.*, $p = 2N(a+d-b-c)/(a+b)$. And, since the sum of the angles at A and B is 90°,

$$\sin^{-1}[2N/(a+b)] + \sin^{-1}[p/(c+d)] + \sin^{-1}[(c-d)/(c+d)] = 90°.$$

This gives the formula in the Question when $N^2 = 12 \cdot 3 = 9 \cdot 4$. When $N^2 = 18 \cdot 2 = 9 \cdot 4$, or when $N^2 = 50 \cdot 2 = 20 \cdot 5$, the three sines are $\frac{1}{3}$, $\frac{11}{15}$, $\frac{5}{13}$.

10860. (H. J. Woodall.)—Trace the curve

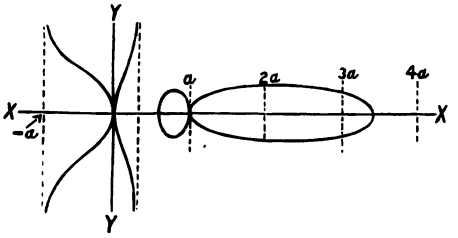
$$(a-x)^3(x^3+y^3)^3 = 4x^2[(a-x)^3+y^2]^3.$$

Solution by T. Thompson.

Take the cube root of each side and separate y^2; then

$$y^2[(a-x)^{\frac{3}{2}} - 4^{\frac{1}{3}}x^{\frac{2}{3}}] = 4^{\frac{1}{3}}x^{\frac{2}{3}}(a-x)^3 - x^2(a-x)^{\frac{3}{2}};$$

therefore $y^2 = \{x^{\frac{2}{3}}(a-x)^{\frac{3}{2}}[4^{\frac{1}{3}}(a-x)^{\frac{1}{2}} - x^{\frac{4}{3}}]\}/[(a-x)^3 - 4^{\frac{1}{3}}x^{\frac{2}{3}}]$.



asymptote asymptote

(i.) Since the equation involves only even powers of y, the curve is symmetrical about the X-axis.

(ii.) Asymptotes parallel to the Y-axis are given by $(a-x)^{\frac{3}{2}} - 4^{\frac{1}{3}}x^{\frac{2}{3}} = 0$, *i.e.*, $(a-x)^3 = 4x^2$, which is equivalent to $3x-a = 0$, $x+a = 0$.

(iii.) The curve intersects the axis of X when $y = 0$, *i.e.*, when $x^2 = 0$, $(a-x)^2 = 0$, $4(a-x)^4 - x^4 = 0$. There are thus double points at (0, 0) and (a, 0), and single points of intersection at $\sqrt{2}(a-x) = \pm x$, *i.e.*, $x = (\sqrt{2}\,a)/(\sqrt{2}\pm 1) = (2\pm\sqrt{2})\,a$. The curve also intersects the X-axis in two imaginary points $2(a-x)^2 + x^2 = 0$.

(iv.) That a real branch may exist y^2 must be positive, and therefore numerator and denominator must have the same sign. Now between $x = -\infty$ and $x = -a$ the numerator is positive and the denominator negative; therefore y is imaginary. Between $x = -a$ and $x = 0$ numerator and denominator are both positive; therefore a real branch exists here. Similarly, a real branch exists between $x = 0$ and $x = \frac{1}{2}a$, y becomes imaginary between $x = \frac{1}{2}a$ and $x = (2-\sqrt{2})\,a$; then there are real branches until $x = (2+\sqrt{2})\,a$, whence the curve is imaginary to $+\infty$.

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(v.) Differentiating the original equation logarithmically, we obtain further details as to slope of the curve. For

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{-[2/(a-x)] + 6x/(x^2+y^2) - 2/x + [6(a-x)]/[(a-x)^2+y^2]}{6y/(x^2+y^2) - 6y/[(a-x)^2+y^2]},$$

since y is a factor of the denominator, the curve always intersects $y = 0$ at right angles (for, when $y = 0$, $dy/dx = \pm\infty$).

This gives us sufficient data to sketch the curve approximately.

15824. (D. Biddle.)—Give instances of quintets of integers in arithmetical progression, four being perfect squares placed in the following order:—(1) the first three and the last, (2) the last three and the first, (3) the first two and the last two, (4) all four together. Example of 1—7^2, 13^2, 17^2, 409, 23^2. [An incomplete answer will be considered better than none at all.]

Solution by Lt.-Col. Allan Cunningham, R.E.

Let the five quantities be denoted by V, W, X, Y, Z, or by v^2, w^2, x^2, y^2, z^2 (when squares); and let δ be their common difference. Here $v^2 = w^2 = x^2 = y^2 = z^2$ is an obvious solution of all the cases; other solutions are dealt with below.

Cases (1) & (2).—These two cases are really one; every solution of either with the five numbers in ascending order being also a solution of the other, with the numbers in descending order.

Step i.—To find three squares v^2, w^2, x^2 in arithmetical progression. Take v', x' such that $v'^2 + x'^2 = w^2$. Then $(v'-x')^2 + (v'+x')^2 = 2w^2$; so that $v = v'-x'$, $x = v'+x'$. The general solution of $v'^2 + x'^2 = w^2$ is known to be given by $v' = t^2-u^2$, $w' = t^2+u^2$, $x' = tu$ (t, u one odd, one even); whence $v = t^2-u^2-2tu$, $w = t^2+u^2$, $x = t^2-u^2+2tu$. Here v^2, w^2, x^2 are in arithmetical progression, with common difference

$$\delta = 4tu(t^2-u^2).$$

Step ii.—Here $Y = (t^2+u^2)^2 + 8tu(t^2-u^2)$, which need not be a square, and $Z = (t^2+u^2)^2 + 12tu(t^2-u^2) = z^2$ (a perfect square). Such equations as this last are known to have an infinite number of solutions (t, u, z) in integers, derivable in succession from the minimum solution (say t_1, u_1, z_1); see Euler's *Elements of Algebra*, trans. Hewlett, 1822, pp. 370–372.

Take $z/u^2 = \zeta$, $t/u = \tau$, and assume $\zeta = (\tau^2+1) + a\tau$. Then

$$\zeta^2 = (\tau^2+1)^2 + 12\tau(\tau^2-1),$$

and also $$= (\tau^2+1)^2 + 2a\tau(\tau^2+1) + a^2\tau^2;$$

therefore $$(2a-12)\tau^3 + a^2\tau^2 + (2a+12)\tau = 0,$$

where two sub-cases arise.

(1) Take $a = 6$: this gives $\tau = -\frac{2}{3}$, $\zeta = -\frac{2}{9}\frac{3}{9}$; whence $t = -2$, $u = 3$ $v = 7$, $w = 13$, $x = -17$, $Y = 409$, $z = -23$.

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(2) Take $a = 6$: this gives $\tau = \frac{1}{2}$, $\zeta = -\frac{23}{6}$; whence $t = 3$, $u = 2$; $v = -7$, $w = 13$, $x = 17$, $y = 409$, $z = -23$.

Thus the first two solutions (7^2, 13^2, 17^2, 409, 23^2) are identical.

Step iii.—To derive further solutions, call the quantities used in the first solution a_1, τ_1, ζ_1, t_1, u_1, ..., and the similar quantities in the second solution a_2, τ_2, ζ_2, t_2, u_2, ... Take $z_2/u_2 = \zeta_2$, $t_2/u_2 = \tau_2$. Then

$$\zeta_2^2 = (\tau_2^2+1)^2 + 12\tau_2(\tau_2^2-1).$$

Now assume $\tau_2 = \tau + \tau_1 = \tau + \frac{1}{2}$ [taking the value of τ_1 in (2) above].

Hence
$$\begin{aligned}\zeta_2^2 &= [(\tau+\tfrac{1}{2})^2+1^2] + 12(\tau+\tfrac{1}{2})[(\tau+\tfrac{1}{2})^2-1] \\ &= \tau^4 + 18\tau^3 + \tfrac{153}{2}\tau^2 + \tfrac{171}{2}\tau + (\tfrac{23}{6})^2.\end{aligned}$$

As this biquadratic has the two end coefficients both perfect squares (1^2 and $23^2/4^2$), it may be rendered a perfect square (ζ_2^2) by the same procedure as before: thus, assume $\zeta = \tau^2 + a_2\tau + \frac{23}{6}$; then

$$\zeta^2 = \tau^4 + 2a_2\tau^3 + (a_2^2 + \tfrac{23}{3})\tau^2 + \tfrac{23}{3}a_2\tau + (\tfrac{23}{6})^2.$$

Equating the values of ζ^2,

$$(2a_2-18)\tau^2 + (a_2^2-58)\tau^2 + \tfrac{1}{2}(23a_2-177)\tau = 0;$$

and two sub-cases arise as before:—

(1) Take $a_2 = 9$: this gives $\tau = -\frac{1}{12}$, $\zeta_2 = 647/46^2$; whence

$$\tau_3 = \tau + \tfrac{1}{2} = \tfrac{23}{12}; \quad t_2 = 39, \quad u_2 = 46;$$

$$v_3 = -4183, \quad w_3 = 3637, \quad x_3 = 2993, \quad Y_3 = 8539440, \quad z_3 = 647.$$

(2) Take $a_2 = \frac{177}{23}$: this gives $\tau = \frac{2047}{1380}$; whence

$$\tau_3 = \tau + \tfrac{1}{2} = \tfrac{1117}{690}, \quad t_2 = 2717, \quad u_2 = 1380;$$

$$v_3 = -2021231, \quad w_3 = 9286489, \quad x_3 = 12976609, \quad Y_3 = x_3^2 + \delta_2^2,$$

where $\delta_2 = 4t_2u_2(t_2^2-u_2^2)$, $z_3 = 18240049$.

Thus three solutions of Cases (1), (2) have been found. By taking $\tau_3 = \tau + \tau_2$ new solutions may be found, and so on in succession; the numbers continually increasing in magnitude.

Case 3.—Here v^2, w^2, X, y^2, z^2 are in arithmetical progression. Hence $v^2 + \mathrm{X} = 2w^2$, $w^2 + y^2 = 2\mathrm{X}$, $\mathrm{X} + z^2 = 2y^2$; whence

$$\mathrm{X} = 2w^2 - v^2 = \tfrac{1}{2}(w^2+y^2) = (2y^2-z^2).$$

Hence
$$v^2 + 3z^2 = (2y)^2, \quad 3v^2 + z^2 = (2w)^2.$$

Now in Collins's *Tract* (quoted below), p. 17 *et seq.*, it is shown that these two equations together involve $v'^2 + z'^2 = y'^2$, $v'^2 + 9z'^2 = w'^2$, a pair of equations which are shown (p. 23) to have no solutions in integers. Hence the only solution (in integers) of this case is the obvious one

$$v^2 = w^2 = x^2 = y^2 = z^2.$$

Case 4.—Here V, W, X, Y should all be squares. The above process applied to making $\mathrm{Y} = (t^2+u^2)^2 + 8tu(t^2-u^2) = y^2$ (a square) leads only to the obvious solution $v^2 = w^2 = x^2 = y^2 = z^2$. This case is known, however, to have no other solution; for proofs of this see Matthew Collins, *Tract on the Possible and Impossible Cases of Quadratic Duplicate Equalities in the Diophantine Analysis*, Dublin, 1858, p. 16, Cor. 3.

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10825. (W. J. C. Sharp, M.A.)—Prove (1), (2), and therefrom deduce (3), (4), where

$$(-1)^m (2\sin A)^m \sin(mA+B+\tfrac{1}{2}m\pi)$$
$$= \sin B - \frac{m}{1}\sin(2A+B) + \frac{m(m-1)}{1.2}\sin(4A+B) - \ldots \quad \ldots (1)$$
$$(-1)^m (2\sin A)^m \cos(mA+B+\tfrac{1}{2}m\pi)$$
$$= \cos B - \frac{m}{1}\cos(2A+B) + \frac{m(m-1)}{1.2}\cos(6A+B) - \ldots \quad \ldots (2);$$
$$\int \sin^m x \sin(m+n)x \,.\, dx, \quad \int \sin^m x \cos(m+n)x \,.\, dx \ldots\ldots (3, 4).$$

Solution by R. J. Whitaker.

Denote series (1) by S, and series (2) by C. Multiply series (1) by i, and add series (2), and the result obtained is

$$C + iS = e^{iB} - \frac{m}{1}e^{2Ai+Bi} + \frac{m(m-1)}{1.2}e^{4Ai+Bi} - \ldots$$
$$= e^{iB}\left[1 - e^{2Ai}\right]^m = (-1)^m e^{mAi+Bi}\left(e^{Ai} - e^{-Ai}\right)^m$$
$$= (-1)^m (2\sin A)^m i^m e^{(mA+B)i}.$$

Now $e^{\frac{1}{2}m\pi i} = i^m$, for

$e^{\frac{1}{2}\pi i} = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi = i$, $e^{\pi i} = -1 = i^2$, $e^{\frac{3}{2}\pi i} = -i = i^3$, $e^{2\pi i} = 1 = i^4$, ...; therefore

$$C + iS = (-1)^m (2\sin A)^m \left[\cos(mA+B+\tfrac{1}{2}m\pi) + i\sin(mA+B+\tfrac{1}{2}m\pi)\right].$$

And, equating real and imaginary parts, we obtain results (1) and (2). Put $A = x - \frac{1}{2}\pi$, $B = nx - \frac{1}{2}\pi$, and (1) becomes

$$(-1)^m 2^m \sin^m x \sin(m+n)x$$
$$= -\left(\cos nx - \frac{m}{1}\cos(n+2)x + \frac{m(m-1)}{1.2}\cos(n+4)x - \ldots\right),$$

whence

$$\int \sin^m x \sin(m+n)x\,dx$$
$$= \frac{(-1)^{m+1}}{2^m}\left(\frac{\sin nx}{n} - \frac{m}{1}\frac{\sin(n+2)x}{n+2} + \frac{m(m-1)}{1.2}\frac{\sin(n+4)x}{n+4} - \ldots\right).$$

And, similarly,

$$\int \sin^m x \cos(m+n)x\,dx$$
$$= \frac{(-1)^{m+1}}{2^m}\left(\frac{\cos nx}{n} - \frac{m}{1}\frac{\cos(n+2)x}{n+2} + \frac{m(m-1)}{1.2}\frac{\cos(n+4)x}{n+4} - \ldots\right).$$

15700. (Professor Nanson.)—Eliminate the variables from

$$L_1/M_1 = L_2/M_2 = \ldots = L_n/M_n$$

where L_p, M_p are linear homogeneous functions of $n-1$ variables.

Solution by the Proposer.

Let $L_p = \Sigma_r a_{rp}x_r$, $M_p = \Sigma_r b_{rp}x_r$, where $p, q = 1, 2, \ldots, n$; $r = 1, 2, \ldots, n-1$, and denote the common value of the fractions in

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the Question by $-\lambda$, so that $\Sigma_r(a_{rp}+\lambda b_{rp})\,x_r) = 0$. Hence, eliminating x, we get $\| a_{rp}+\lambda b_{rp} \| = 0$, n equations of order $n-1$ in λ. Denote these by $\Sigma_q c_{pq}\lambda^{q-1} = 0$; then the required eliminant is $| c_{pq} | = 0$, and c_{pq} is the determinant found by omitting the p-th column in the array (a_{rp}), whilst c_{pq} is the sum of the results found from c_{p_1} by replacing $q-1$ of the "a" columns by the corresponding columns from the array (b_{rp}).

15849. (Professor NEUBERG.)—On sait que les pieds M, M′, M″, M‴ des normales abaissées d'un point P sur une conique Σ sont sur une hyperbole équilatère Σ' passant par P et par le centre C de Σ. Etant donnés l'hyperbole équilatère $\Sigma'(xy = 1)$ et les trois points M, M′, M″ sur cette courbe, trouver (1) la relation entre les abscisses des points M, M′, M″, M‴; (2) la relation entre les abscisses des points M, M′, M″, P; (3) celle entre les abscisses des points M, M′, M″, C.

Solution by the PROPOSER.

Prenons pour Σ l'ellipse $a^2y^2+b^2x^2-a^2b^2 = 0$ et soient (α, β) les coordonnées de P; l'équation de Σ' sera $c^2xy+b^2\beta x-a^2\alpha y = 0$. Le centre O de Σ' a pour coordonnées $\alpha_1 = a^2\alpha/c^2$, $\beta_1 = -b^2\beta/c^2$. Transportons les axes coordonnés au point O; en remplaçant x, y par $x+\alpha_1$, $y+\beta_1$, on trouve pour les nouvelles équations de Σ et Σ':

$$a^2(y+\beta_1)^2+b^2(x+\alpha_1)^2 = a^2b^2, \quad xy = \alpha_1\beta_1 \ldots\ldots\ldots\ldots (1)$$

les nouvelles coordonnées des points C et P sont

$$\text{C}: \quad \alpha'_1 = -\alpha_1, \quad \beta'_1 = -\beta_1;$$

$$\text{P}: \quad \alpha_2 = \alpha-\alpha_1 = -b^2\alpha/c^2 = b^2\alpha'_1/a^2, \quad \beta_2 = a^2\beta/c^2 = a^2\beta'_1/b^2 \ldots (2).$$

Soient (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) les coordonnées des points M, M′, M″, M‴ par rapport aux asymptotes de Σ'. En éliminant y entre les équations (1), on trouve

$$b^2x^4+2b^2\alpha_1x^3+(a^2\beta_1^2+b^2\alpha_1^2-a^2b^2)x^2+2a^2\alpha_1\beta_1^2x+a^2\alpha_1^2\beta_1^2 = 0;$$

d'où l'on conclut

$$\Sigma x_1 = -2\alpha_1 = 2\alpha'_1, \qquad \Sigma x_1x_2 = (a^2\beta_1^2+b^2\alpha_1^2-a^2b^2)/b^2,$$
$$\Sigma 1/x_1 = -2/\alpha_1 = 2/\alpha'_1, \quad x_1x_2x_3x_4 = a^2\alpha_1^2\beta_1^2/b^2.$$

Par analogie, $\Sigma y_1 = 2\beta'_1$, $\qquad \Sigma y_1y_2 = (a^2\beta_1^2+b^2\alpha_1^2-a^2b^2)/a^2$,

$$\Sigma 1/y_1 = 2/\beta'_1, \quad y_1y_2y_3y_4 = b^2\alpha_1^2\beta_1^2/a^2.$$

(1) La relation cherchée entre les abscisses des points M, M′, M″, M‴ est $\Sigma x_1 \Sigma 1/x_1 = 4$, ou $\Sigma x_1 \Sigma x_1x_2x_3 = 4x_1x_2x_3x_4$; elle est du second degré en x_4.

(3) Les équations $\alpha'_1 = \frac{1}{2}\Sigma x_1$, $\beta'_1 = \frac{1}{2}\Sigma y_1$ expriment les relations entre C et les quatre points M, M′, M″, M‴. En effet, si N est le centre des moyennes distances du quadrangle MM′M″M‴, C et O sont symétriques par rapport à N ou diamétralement opposés sur la conique des neuf points de ce quadrangle. Ceci donne le moyen de construire M‴ et C au moyen de M, M′, M″ et Σ'. Car, soit G le barycentre du triangle

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MM'M''; désignons par N' le point qui divise dans le rapport 1:3 la distance de G à un point quelconque m de Σ' et par C' le symétrique de O par rapport à N'. Le lieu de N' est une courbe Σ'' homothétique à Σ' par rapport à G et le lieu de C' est une courbe Σ''' homothétique à Σ'' par rapport à O. Le point C est l'un des deux points de rencontre des deux hyperboles Σ', Σ''', dont les asymptotes sont parallèles. De C on déduit N, et de N, M'''.

(2) D'après les formules (2), les coordonnées de P se déduisent de celles de C si l'on connait le rapport a^2/b^2; or, nos formules donnent

$$\Sigma x_1x_2/\Sigma y_1y_2 = a^2/b^2, \quad x_1x_2x_3x_4/y_1y_2y_3y_4 = a^4/b^4.$$

Pour avoir a^2 et b^2, observons que

$$\Sigma x_1x_2 = (a^2/b^2)\beta_1^2 + a_1^2 - a^2, \quad \Sigma y_1y_2 = \beta_1^2 + (b^2/a^2, a_1^2 - b^2.$$

15864. (Robt. W. D. Christie.)—Prove that the Pellian equation $X_n^2 - PY_n^2 = 1$ can be put into a trigonometrical form, and give the general solution. *E.g.*,

$$X_3 = \cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x,$$
$$Y_3 = \sin 3\theta = (4\cos^2\theta - 1)\sin\theta = 4x^2y - y,$$

x and y having integral or fractional values.

Solution by the Proposer.

Let $X_n^2 - PY_n^2 = 1$; then

$$X = \frac{2^{n-1}}{0!}x^n - \frac{2^{n-3}n}{1!}x^{n-2} + \frac{2^{n-5}n.n-3}{2!}x^{n-4} - \frac{2^{n-7}n.n-4.n-5}{3!}x^{n-6} + \ldots$$

(the series continues till the results are not negative)

$$= \Sigma_0^m \frac{2^{n-2m-1}n(n-m)!}{m!\,(n-2m\ !}x^{n-2m},$$

$$Y = \frac{2^{n-1}}{0!}x^{n-1}y - \frac{2^{n-3}n-2}{1!}x^{n-3}y + \frac{2^{n-5}n-3.n-4}{2!}x^{n-5}y - \ldots$$

$$= \Sigma_0^m \frac{2^{n-2m-1}(n-m-1)!}{m!\,(n-2m-1)!}x^{n-2m-1}y.$$

These values will be found to coincide exactly with the values of $X = \cos n\theta$ and $Y = \sin n\theta$, when $\cos n\theta$ is developed into the form $\cos^n\theta$ and $\sin n\theta$ into the form $\sin\theta(\cos^n\theta)$, *i.e.*,

$$\sin n\theta = \sin\theta\left\{(2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3} + \frac{n-3.n-4}{2!}(2\cos\theta)^{n-5} - \ldots\right\};$$

e.g., $n = 3$ or 5 gives us

$$X_3 = 4x^3 - 3x, \qquad X_5 = 16x^5 - 20x^3 + 5x,$$
$$Y_3 = 4x^2y - y, \qquad Y_5 = 16x^4y - 12x^2y + y.$$

Values of X and Y, being trigonometrical, may be put in various other ways.

[These forms, though somewhat clumsy, are believed to be new, and have been extracted from an old note book of 1895 and have appeared in the *Reprint* for another purpose.]

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15837. (Professor SANJÁNA, M.A.)—Points P, Q, R are taken in the sides BC, CA, AB of a triangle, such that the areas AQR, BRP, CPQ are each equal to a given area T. Show how to construct for the points, and prove that the distances of any two from the sides not containing them are together equal to the corresponding altitude of the triangle.

Solution by A. M. NESBITT, M.A.

I shall take the latter part first, and shall show that, if the sum of the distances of Q and R from BC is equal to the distance of A from BC, and also the sum of the distances of R and P from AC is equal to the distance of B from AC, then the three triangles will be equal.

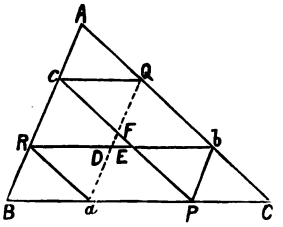

Fig. 1.

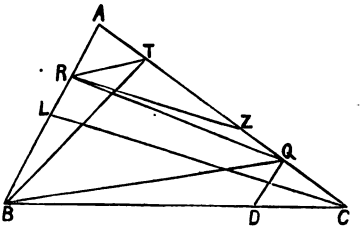

Fig. 2.

Take any point P in BC and draw Pb, Pc parallel to the sides AB, AC; also draw bR, cQ parallel to BC, and Ra parallel to AC. Then clearly Ba = PC = cQ; therefore, if Qa be joined, it is parallel to AB. Also BRa = PbC,

$$\text{PR}a = \tfrac{1}{2}\text{ parallelogram PER}a = \tfrac{1}{2}\text{ parallelogram P}a\text{D}b$$
$$= \tfrac{1}{2}\text{ parallelogram P}b\text{QF} = \text{PQ}b;$$

therefore BRP = CPQ = AQR by symmetry. And the altitudes have plainly the property stated.

In AC (Fig. 2) take T so that ABT may be equal to the given area T; then, since the triangles BQR, BRP, CPQ are all equal, and the least area of an inscribed triangle such as PQR is a quarter of the original triangle, it follows that $4AT \leqslant AC$. Take $CZ^2 = AC^2 - 4AT.AC$. Draw ZR parallel to CL, L being the mid-point of AB. Draw BQ parallel to TR, QD parallel to AB, and join QR. Then the triangle AQR is clearly equal to ABT; it remains to show that Q and R satisfy the altitude property, *i.e.*, that QD = AR.

Now $LR/AL = CZ/AC$; therefore $(LR/AL)^2 = (AC-4AT)/AC$, or $4AT/AC = 1 - LR^2/AL^2 = (BR.RA)/AL^2$ or $AT/AC = BR.RA/AB^2$. Hence $AT/AR = BR.AC/AB^2 = AQ/AB$ by parallels; therefore

$$BR/AQ = AB/AC = AR/CQ,$$

by subtracting numerator and denominator; therefore

$$AR/CQ = AB/AC = QD/CQ;$$

therefore

$$AR = QD.$$

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